Trading Networks with Bilateral Contracts*

Tamás Fleiner†
Budapest University of Technology and Economics

Zsuzsanna Jankó‡
Corvinus University

Akihisa Tamura§
Keio University

Alexander Teytelboym¶
University of Oxford

November 8, 2016

Abstract

We consider general networks of bilateral contracts that include supply chains. We define a new stability concept, called *trail stability*, and show that any network of bilateral contracts has a trail-stable outcome whenever agents' choice functions satisfy full substitutability. Trail stability is a natural extension of chain stability, but is a stronger solution concept in general contract networks. Trail-stable outcomes are not immune to deviations of arbitrary sets of firms. In fact, we show that outcomes satisfying an even more demanding stability property – *full trail stability* – always exist. For fully trail-stable outcomes, we prove results on the lattice structure, the rural hospitals theorem, strategy-proofness and comparative statics of firm entry and exit. We pin down a condition under which trail-stable and fully trail-stable outcomes coincide. We then completely describe the relationships between various other concepts. When contracts specify trades and prices, we also show that competitive equilibrium exists in networked markets even in the absence of fully transferrable utility. The competitive equilibrium outcome is (fully) trail-stable.

*This paper unites three independent works *Trading networks with bilateral contracts* circulated by Teytelboym in early 2014 (a much earlier draft that appeared in his doctoral thesis in 2013), Jankó's Master's thesis *Generalized stable matchings: theory and applications*, supervised by Fleiner and completed, but not widely circulated, in 2011, as well as *Stability of generalized network flows* by Fleiner, Jankó, and Tamura, completed, but not widely circulated, in 2014. We would like to thank Samson Alva, Scott Kominers, and Michael Ostrovsky for their valuable comments on the recent versions of the paper. Vincent Crawford, Umut Dur, Jens Gudmundsson, Claudia Herrestahl, Paul Klemperer, Collin Raymond, and Zaifu Yang also gave great comments on much earlier drafts. We had enlightening conversations with Alex Nichifor, Alex Westkamp and M. Bumin Yenmez about the project. Moreover, we are also grateful to seminar participants at the Southern Methodist University, National University of Singapore, ECARES, CIREQ Matching Conference (Montréal), Workshop on Coalitions and Networks (Montréal), the 12th and the 13th Meetings of the Society of Social Choice and Welfare, the 3rd International Workshop on Matching Under Preferences (Glasgow) and AMMA (Chicago) for their comments.

†Research was supported by the OTKA K108383 research project and the MTA-ELTE Egerváry Research Group. Part of the research was carried out during two working visits at Keio University. E-mail: fleiner@cs.bme.hu

‡Research was supported by the OTKA K109240 research project and the MTA-ELTE Egerváry Research Group. E-mail: jzsuzsy@cs.elte.hu

§Research was supported by Grants-in-Aid for Scientific Research (B) from JSPS. E-mail: aki-tamura@math.keio.ac.jp

¶Institute for New Economic Thinking at the Oxford Martin School. Email: alexander.teytelboym@inet.ox.ac.uk

1 Introduction

Modern production is highly interconnected and many firms have a large numbers of buyers and suppliers. In this paper, we study the structure of contract relationships between firms. In our model, firms have heterogeneous preferences over sets of bilateral contracts with other firms. Contracts may encode many dimensions of a relationship including the quantity of a good traded, time of delivery, quality, and price. The universe of possible relationships between firms is described by a *contract network* – a multi-sided matching market in which firms form downstream contracts to sell outputs and upstream contracts to buy inputs.

We focus on the existence and structure of stable outcomes in decentralized, real-world matching markets. In production networks that we consider in this paper, stable outcomes play the role of an equilibrium concept and may serve as a reasonable prediction of the outcome of market interactions (Fox, 2010).[1] We find a general result: any contract network has an outcome that satisfies a natural extension of *pairwise stability* (Gale and Shapley, 1962). Our model of matching markets subsumes many previous models of matching with contracts, including many-to-one (Gale and Shapley, 1962, Crawford and Knoer, 1981, Kelso and Crawford, 1982, Hatfield and Milgrom, 2005) and many-to-many matching markets (Roth, 1984, Sotomayor, 1999, 2004, Echenique and Oviedo, 2006, Klaus and Walzl, 2009).

We build on a seminal contribution by Ostrovsky (2008), who introduced a matching model of *supply chains*. In a supply chain, there are agents, who only supply inputs (e.g. farmers); agents, who only buy final outputs (e.g. consumers); while the rest of the agents are intermediaries, who buy inputs and sell outputs (e.g. supermarkets). All agents are partially ordered along the supply chain: downstream (upstream) firms cannot sell to (buy from) firms upstream (downstream) i.e. the contract network is *acyclic*. His key assumption about the market, which we retain in his paper, was that firms' choice functions over contracts satisfy *same-side substitutability* and *cross-side complementarity* conditions (Hatfield and Kominers (2012) later called these conditions *full substitutability*). This assumption requires that firms view any downstream or any upstream contracts as substitutes, but any downstream and any upstream contract as complements.[2] Ostrovsky (2008) showed that any supply chain

[1]The "market design" literature has emphasized the importance of the existence of stable outcomes in order to prevent centralized matching markets from unraveling (Roth, 1991). While much of this paper is inspired by this line of research, we do not focus on practical market design applications here.

[2]Same-side substitutability is a fairly strong assumption as, for example, it rules out any complementarities in inputs. There is evidence that modern manufacturing firms rely on many complementary inputs (Milgrom and Roberts, 1990, Fox, 2010). Hatfield and Kominers (2015b) consider a multilateral matching market with complements when utility is quasilinear in prices. Alva and Teytelboym (2015) analyze supply chains in which inputs could be complementary or substitutable with general preferences. While production decisions often create externalities, we assume that firms only care about the contracts they are involved in (Sasaki and Toda, 1996, Bando, 2012, Pycia, 2012, Pycia and Yenmez, 2015). In addition, we work in a

has a *chain-stable* outcome for which there are no blocking downstream chains of contracts. Hatfield and Kominers (2012) further showed that, in the presence of network acyclicity, chain-stable outcomes are equivalent to (what we call) *set-stable* outcomes i.e. those that are immune to deviations by arbitrary sets of firms. Even under full substitutability, chain-stable/set-stable outcomes in general supply chains may be Pareto inefficient.[3]

While a supply chain may be a good model of production in certain industries (Antràs and Chor, 2013), in general, firms simultaneously supply inputs to *and* buy outputs from other firms (possibly through intermediaries). If this is the case, we say a contract network contains a contract *cycle*. For example, the sectoral input-output network of the U.S. economy, illustrated by Acemoglu et al. (2012, Figure 3), shows that American firms are very interdependent and the contract network contains many cycles. Consider a coal mine that supplies coal to a steel factory. The factory uses coal to produce steel, which is an input for a manufacturing firm that sells mining equipment back to the mine. This creates a contract cycle. However, Hatfield and Kominers (2012) showed that if a contract network has a contract cycle then set-stable outcomes may fail to exist. Our first result shows that checking whether an outcome is in fact set-stable is computationally hard. We then show that, even in the presence of contract cycles, outcomes that satisfy a weaker notion of stability – *trail stability* – can still be found. A trail of contracts is a sequence of distinct contracts in which the buyer in one contract is the seller in the subsequent one. We argue that trail stability is a useful, natural and intuitive equilibrium concept for the analysis of matching markets in networks. Along a blocking trail, firms make unilateral offers to their neighbors while conditionally accepting a sequence of previous pairwise blocks. Firms can receive several offers along the trail. Trail-stable outcomes rule out any sequence of such consecutive pairwise blocks. Trail stability is equivalent to chain stability (and therefore to set stability under our assumptions) in acyclic contract networks and to pairwise stability in two-sided many-to-many matching markets with contracts. Unsurprisingly, therefore, trail-stable outcomes may also be Pareto inefficient (Blair, 1988).

In order to analyze properties of trail-stable outcomes, we introduce another stability notion, called *full trail stability*, which does not force intermediary firms to accept all the contracts along a trail, but rather only sign upstream/downstream pairs as they are offered along the trail. We argue that this could also be seen as a useful stability notion for short-

---

complete information environment; for a treatment with asymmetric information, see Roth (1989), Ehlers and Massó (2007), Chakraborty et al. (2010). Extending our model to incorporate incomplete information and externalities is a promising area for further research.

[3]Inefficiency arises even in two-sided many-to-many matching markets without contracts: Blair (1988) and Roth and Sotomayor (1990, p. 177) provide the earliest examples; Echenique and Oviedo (2006), Klaus and Walzl (2009) discuss the setting with contracts. Westkamp (2010) provides necessary and sufficient conditions on the structure contract relationships in the supply chain for chain-stable outcomes to be efficient.

run contract relationships. But studying full trail stability also allows us to use familiar fixed-point theorems and other techniques from the matching literature. Fully trail-stable outcomes correspond to the fixed-points of an operator and form a particular lattice structure for *terminal agents*, who can sign only upstream or only downstream contracts. The lattice reflects the classic opposition-of-interests property of two-sided markets, but in our case it is between terminal buyers and terminal sellers. In addition to this strong lattice property, we extend previous results on the existence of buyer- and seller-optimal stable outcomes, the rural hospitals theorem, strategy-proofness as well as comparative statics on firm entry and exit that have only been studied in a supply-chain or two-sided setting under general choice functions (see Figure 2). Fully trail-stable and trail-stable coincide under *separability*, a condition that ensures that decisions over certain pairs of upstream and downstream contracts are taken independently from others. We provide a complete description of the relationships between all stability notions – set stability, chain stability, trail stability, full trail stability – that we use in this paper.

Our work complements a recent paper by Hatfield et al. (2015) on the properties of set-stable outcomes in general contract networks. They show that in general contract networks, under certain conditions, set-stable outcomes coincide with (what we call) *strongly trail-stable* outcomes i.e. those immune to coordinated deviations by a set of firms which is simultaneously signing a trail of contracts. Our paper is also related to the stability of (continuous and discrete) network flows discussed by Fleiner (2009, 2014). In these models, agents choose the amount of "flow" they receive from upstream and downstream agents and have preferences over who they receive the "flow" from. The network flow model allows for cycles. However, the choice functions in the network flow models are restricted by Kirchhoff's (current) law (the total amount of incoming (current) flow is equal to the total amount of outgoing flow) and in the discrete case, these choice functions are special cases of Ostrovsky (2008). This paper therefore generalizes both of the supply chain and the network flow models, while offering two appealing new stability concepts.

We also consider a setting in which every contract specifies a trade and a price. We ask whether there is a competitive equilibrium outcome: a vector of prices at which agents demand precisely the trades which are realized. However, to specify competitive equilibrium fully, we also need to find prices for trades that are not realized. We find these prices constructively by adapting the salary-adjustment process of Kelso and Crawford (1982) and Roth (1984). While Hatfield et al. (2013) and Hatfield and Kominers (2015b) also considered the existence of competitive equilibrium in general contract networks, in addition to the assumptions in this paper, they also assumed that firms' profit functions are *quasilinear* in a continuous numeraire (i.e. there is transferrable utility). These assumptions not only guarantee the existence, but also efficiency and stability of competitive equilibrium. However, the quasilinearity of the firms' profit function is a strong assumption in many settings. Several reasons for the failure of this assumption can be found in the empirical literature. First, firms may have financing constraints since access to debt and equity financing differs across firms (Fazzari et al., 1988). Second, firm management may exhibit a version of the "wealth effect" by investing free cash flow into wasteful investments (Jensen, 1986). Finally, there is evidence that in volatile markets firms are risk-averse (Frank, 1990). Hatfield et al. (2013, p. 18) point out that:

> for contractual sets that allow for continuous transfers, in the presence of quasilinearity, supply chain structure is not necessary for the existence of stable outcomes, although full substitutability is. It is an open question why the presence of a continuous numeraire can replace the assumption of a supply chain structure in ensuring the existence of stable outcomes.

We dispense with quasilinearity entirely while retaining a general network structure. We establish the existence of competitive equilibrium in networked markets without transferrable utility under two extra mild conditions.[4] The competitive equilibrium outcomes are also trail-stable.

Figure 1 summarizes stability concepts, results and nomenclature used in this paper with reference to previous work, while Figure 2 describes which previous results we have generalized in our setting of trading network with general choice functions. We proceed as follows. In Section 2, we present the ingredients of the model, including the contract network, restrictions on firms choices of contracts, and various stability concepts. In Section 2.5, we show that set stability is computationally intractable in general contract networks. Then, in Section 3, we state our key result of trail-stable outcomes in general contract networks. We then introduce full trail stability and dig deeper into the structure of fully trail-stable outcomes in Section 3.1. We conclude this section by describing overall relationship between different stability notions. In Section 4, we show how to construct competitive equilibrium allocations in a model with prices. Finally, we conclude and outline some directions for future work. The proofs are in the Appendix.

[4] There has been relatively little work on the existence of competitive equilibrium with indivisible goods without transferrable utility, except in one-to-one markets (Quinzii, 1984, Gale, 1984, Demange and Gale, 1985, Alaei et al., 2011, Morimoto and Serizawa, 2015, Herings, 2015).

	General networks	General choice functions	Competitive equilibrium	Existence and structure	New stability concepts used	Corresponding name in this paper
Ostrovsky (2008)	✗ acyclic	✓	✗	✓	Chain-stable	Path-stable
Westkamp (2010)	✗ acyclic	✓	✗	✓	Group-stable or Setwise-stable, Core	–
Hatfield and Kominers (2012)	✗ acyclic	✓	✗	✓	Stable or Weak Setwise-stable	Set-stable
Hatfield et al. (2013), Hatfield and Kominers (2015b)	✓	✗ quasilinear	✓	✓	Strong group-stable	–
Hatfield et al. (2015)	✓	✓	✓	✗	Chain-stable	Strong trail-stable
This paper	✓	✓	✓	✓	Trail-stable, Fully trail-stable	Trail-stable, Fully trail-stable

Figure 1: Relationship to previous work.

2 Model

2.1 Ingredients

There is finite set of agents (firms or consumers) F and a finite set of contracts (contract network) X.[5] A contract $x \in X$ is a bilateral agreement between a buyer $b(x) \in F$ and a seller $s(x) \in F$. A *(trading) cycle* in X is a sequence of firms $(f_1, \ldots, f_M)$ such that for all $m \in \{1, \ldots, M\}$ there exists a contract x_m such that $s(x_m) = f_m$ and $b(x_m) = f_{m+1}$ (subscripts modulo M). Hence, $F(x) := \{s(x), b(x)\}$ is the set of firms associated with contract x and, more generally, $F(Y)$ is the set of firms associated with contract set $Y \subseteq X$.

[5]The standard justification for this assumption is given by Roth (1984, p. 49): "elements of a [contract] can take on only discrete values; salary cannot be specified more precisely than to the nearest penny, hours to the nearest second, etc." In fact, the finiteness assumption is not necessary for our proofs. We only require that the set of contracts between any two agents forms a lattice.

Paper	Theorem	Description	Generalization in this paper
Ostrovsky (2008)	Theorem 1	Existence of stable outcomes	Theorem 2 and Lemma 2
Ostrovsky (2008); Hatfield and Kominers (2012)	Theorem 2; Theorem 4	Buyer- and seller-optimality	Lemma 3 and Lemma 4
Hatfield and Kominers (2012)	Theorem 8	Rural hospitals theorem	Proposition 1
Hatfield and Kominers (2012)	Theorem 10	Strategy-proofness	Proposition 2
Ostrovsky (2008)	Theorem 3	Firm entry	Proposition 3
Hatfield and Kominers (2013)	Theorem	Vacancy chain dynamics	Proposition 4
Hatfield et al. (2013)	Theorem 1	Competitive equilibrium	Theorem 3

Figure 2: Previous results generalized in this paper to a trading network setting with general choice functions.

Call $X_f^B := \{x \in X | b(x) = f\}$ and $X_f^S := \{x \in X | s(x) = f\}$ the sets of f's upstream and downstream contracts – for which f is a buyer and a seller, respectively. Clearly, Y_f^B and Y_f^S form a partition over the set of contracts $Y_f := \{y \in Y | f \in F(y)\}$ which involve f, since an agent cannot be a buyer and a seller in the same contract.

We can show graphically that our structure is more general than the setting described by Ostrovsky (2008), Westkamp (2010) or Hatfield and Kominers (2012). Each firm $f \in F$ is associated with a vertex of a directed multigraph (F, X) and each contract $x \in X$ is a directed edge of this graph. For any f, X_f^B is represented by a set of incoming edges and X_f^S is represented by outgoing edges. In Figure 3, we illustrate a three-level supply chain with two producers, two intermediaries and two final consumers. A *acyclic* contract network (a *supply chain*) contains no trading cycles. Supply chains require a partial order on the firms' positions in the chain although firms may sell to (buy from) any downstream (upstream) level. Hence, in Figure 3, the right producer sells directly to the left consumer bypassing the intermediary. In our model, we consider general contract networks, which may contain contract cycles (i.e. directed cycles on the graph), illustrated in Figure 4.

Every firm has a choice function C^f, such that $C^f(Y_f) \subseteq Y_f$ for any $Y_f \subseteq X_f$.[6] We say that choice functions of $f \in F$ satisfy the *irrelevance of rejected contracts (IRC)* condition if for any $Y \subseteq X$ and $C^f(Y) \subseteq Z \subseteq Y$ we have that $C^f(Z) = C^f(Y)$ (Blair, 1988, Alkan, 2002, Fleiner, 2003, Echenique, 2007, Aygün and Sönmez, 2013).[7]

[6]Since firms only care about their own contracts, we can write $C^f(Y)$ to mean $C^f(Y_f)$.

[7]In our setting IRC is equivalent to the Weak Axiom of Revealed Preference (Alva, 2015b).

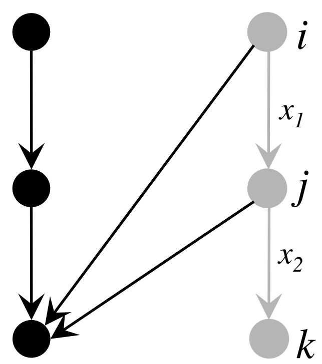

Figure 3: Supply chain

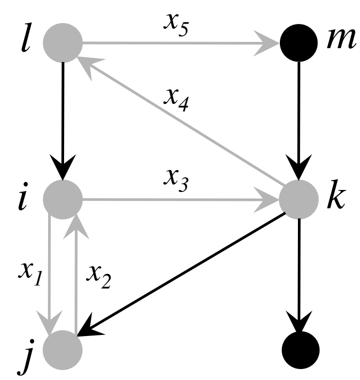

Figure 4: Contract network

For any $Y \subseteq X$ and $Z \subseteq X$, define the *chosen* set of upstream contracts

$$C_B^f(Y|Z) := C^f(Y_f^B \cup Z_f^S) \cap X_f^B \tag{2.1}$$

which is the set of contracts f chooses as a buyer when f has access to upstream contracts Y and downstream contracts Z. Analogously, define the chosen set of downstream contracts

$$C_S^f(Z|Y) := C^f(Z_f^S \cup Y_f^B) \cap X_f^S \tag{2.2}$$

Hence, we can define *rejected* sets of contracts $R_B^f(Y|Z) := Y_f \setminus C_B^f(Y|Z)$ and $R_S^f(Z|Y) := Z_f \setminus C_S^f(Z|Y)$. An *outcome* $A \subseteq X$ is a set of contracts.

A set of contracts $A \subseteq X$ is *individually rational* for an agent $f \in F$ if $C^f(A_f) = A_f$. We call set A *acceptable* if A is individually rational for all agents $f \in F$. For sets of contracts $W, A \subseteq X$, we say that A is (W, f)-*rational* if $A_f \subseteq C^f(W_f \cup A_f)$ i.e. if the agent f chooses all contracts from set A_f whenever she is offered A alongside W. Set of contracts A is W-*rational* if A is (W, f)-rational for all agents $f \in F$. Note that contract set A is individually rational for agent f if and only if it is $(\emptyset, f)$-rational. If $y \in X_f^B$ and $z \in X_f^S$ then $\{y, z\}$ is a (W, f)-*rational pair* if neither y nor z is (W, f)-rational but $\{y, z\}$ is (W, f)-rational. Note that any rational pair consists of exactly one upstream and one downstream contract.

2.2 Assumptions on choice functions

We can now state our key assumption on choice functions introduced by Ostrovsky (2008).

Definition 1. Choice functions of $f \in F$ are *fully substitutable* if for all $Y' \subseteq Y \subseteq X$ and $Z' \subseteq Z \subseteq X$ they are:

1. *Same-side substitutable* (SSS):

   (a) $R_B^f(Y'|Z) \subseteq R_B^f(Y|Z)$

   (b) $R_S^f(Z'|Y) \subseteq R_S^f(Z|Y)$

2. *Cross-side complementary* (CSC):

   (a) $R_B^f(Y|Z) \subseteq R_B^f(Y|Z')$

   (b) $R_S^f(Z|Y) \subseteq R_S^f(Z|Y')$

Contracts are fully substitutable if every firm regards any of its upstream or any of its downstream contracts as substitutes, but its upstream and downstream contracts as complements. Hence, rejected downstream (upstream) contracts continue to be rejected whenever the set of offered downstream (upstream) contracts expands or whenever the set of offered upstream (downstream) contracts shrinks.

We also introduce a new restriction on choice functions that will play a major role in linking together various stability concepts described in this paper.

Definition 2. Choice functions of $f \in F$ are *separable* if for any $A, W \subseteq X$ and $y \in X_f^B \setminus A$ and $z \in X_f^S \setminus A$, whenever A is (W, f)-rational, and $\{y, z\}$ is a (W, f)-rational pair, then $A \cup \{y, z\}$ is (W, f)-rational.

Separable choice functions impose a kind of independence on choices of pairs of upstream and downstream contracts. It says that whenever the firm chooses A alongside some set W and $\{y, z\}$ alongside W (but y and z would not be chosen separately alongside W since $\{y, z\}$ is a (W, f)-rational pair), then it would choose $A \cup \{y, z\}$ alongside W. Suppose signing A and $\{y, z\}$ are decisions made by separate units of the firm. Separable choice functions say that it can delegate the joint input-output decisions to the units because its overall choices do not require any coordination between the units. One natural example of separable choice functions is the following: suppose each firms totally orders individual upstream contracts and individual downstream contracts. Whenever a firm is offered k downstream and l upstream contracts, it chooses the z best upstream and the z best downstream contracts where $z = \min(k, l)$.

2.3 Laws of Aggregate Demand and Supply

We first re-state the familiar Laws of Aggregate Demand and Supply (LAD/LAS) (Hatfield and Milgrom, 2005, Hatfield and Kominers, 2012). LAD (LAS) states that when a firm has more upstream (downstream) contracts available (holding the same downstream (upstream) contracts), the number of downstream (upstream) contracts the firms chooses does not increase more than the number of upstream (downstream) contracts the firm chooses. Intuitively, an increase in the availability of contracts on one side, does not increase the difference between the number of contracts signed on either side.

Definition 3. Choice functions of $f \in F$ satisfy the Law of Aggregate Demand if for all $Y, Z \subseteq X$ and $Y' \subseteq Y$

$$|C_B^f(Y|Z)| - |C_B^f(Y'|Z)| \geq |C_S^f(Z|Y)| - |C_S^f(Z|Y')|$$

and the Law of Aggregate Supply if for all $Y, Z \subseteq X$ and $Z' \subseteq Z$

$$|C_S^f(Z|Y)| - |C_S^f(Z'|Y)| \geq |C_B^f(Y|Z)| - |C_B^f(Y|Z')|$$

We can easily show that LAD/LAS imply IRC, extending the result by Aygün and Sönmez (2013).

Lemma 1. In any contract network X if choice functions of $f \in F$ satisfy full substitutability and LAD/LAS then the choice functions of f satisfy IRC.

2.4 Terminal agents and terminal superiority

We now introduce some terminology that describes contracts of agents, who only act as buyers or only act as sellers. A firm f is a *terminal seller* if there are no upstream contracts for f in the network and f is a *terminal buyer* if the network does not contain any downstream contracts for f. An agent who is either a terminal buyer or a terminal seller is called a *terminal agent*. Let $\mathcal{T}$ denote the set of terminal agents in F and for a set A of contracts let us denote the *terminal contracts of* A by $A_{\mathcal{T}} := \bigcup\{A_f | f \in \mathcal{T}\}$. A set Y of contracts is *terminal-acceptable* if there is an acceptable set A of contracts such that $Y = A_{\mathcal{T}}$. If A and W are terminal-acceptable sets of contracts then we say that A is *seller-superior* to W (denoted by $A \succeq^S W$) if $C_f(A_f \cup W_f) = A_f$ for each terminal seller f and $C_g(A_g \cup W_g) = W_g$ for each terminal buyer g. Similarly, A is *buyer-superior* to W (denoted by $A \succeq^B W$) if $C_f(A_f \cup W_f) = W_f$ for each terminal seller f and $C_g(A_g \cup W_g) = A_g$ for each terminal buyer g. Clearly, these relations are opposite, that is, $W \succeq^S A$ if and only if $A \succeq^B W$ holds. Whenever either relation holds, we call this partial order on outcomes *terminal superiority*. Terminal agents are going to play a key role when we describe the structure of outcomes in contract networks.

2.5 Stability concepts

We start off by defining two stability notions that have appeared in previous work.

Definition 4. An outcome $A \subseteq X$ is *set-stable*[8] if:

[8] Klaus and Walzl (2009) call set-stable outcomes "weak setwise stable" and Hatfield and Kominers (2012)

1. A is acceptable.

2. There exist no non-empty blocking set of contracts $Z \subseteq X$, such that $Z \cap A = \emptyset$ and Z is (A, f)-rational for all $f \in F(Z)$.

Set-stable outcome are immune to deviations by *sets* of firms, which can re-contract freely among themselves while keeping any of their existing contracts. Set-stable outcomes always exist in acyclic networks if choice functions are fully substitutable. In order to study more general contract networks, we first introduce trails of contracts.

Definition 5. A non-empty sequence of different contracts $T = \{x_1, \ldots, x_M\}$ is a *trail* if $b(x_m) = s(x_{m+1})$ holds for all $m = 1, \ldots, M-1$.

While a trail may not contain the same contract more than once, it may include the same agents any number of times. Figure 4 illustrates a trail that starts from firm i to firm j via firm k. A trail T is a *path* if all the agents $F(T)$ involved in the trail are distinct. A path from firm i to firm j is illustrated in Figure 3.

Definition 6. An outcome $A \subseteq X$ is *strongly trail-stable* if

1. A is acceptable.

2. There is no trail T, such that $T \cap A = \emptyset$ and T is (A, f)-rational for all $f \in F(T)$.

Hatfield et al. (2015) showed that in general contract networks set-stable outcomes are equivalent to strongly trail-stable outcomes whenever choice functions satisfy full substitutability and Laws of Aggregate Demand and Supply.[9] However, Fleiner (2009) and Hatfield and Kominers (2012) showed that a set-stable outcomes may not exist in general contract networks (see Example 1 below). Moreover, our first result demonstrates that set stability is computationally intractable.[10] Let us define decision problem GS as follows. An instance of

call them "stable": we take the middle ground. Westkamp (2010) applies the label "group stable" to "setwise stable outcomes" (Sotomayor, 1999, Echenique and Oviedo, 2006, Klaus and Walzl, 2009).

[9] Hatfield et al. (2015) call trails "chains" and strong trail stability "chain stability". We use our terminology to avoid the confusion with the original definition of "chains" and "chain stability" in Ostrovsky (2008). Our distinction between "trails" and "paths" is used in most graph theory textbooks.

[10] Powerful solution concepts is economics often derive their appeal for normative reasons: they have sensible properties or make intuitive sense in particular settings. In fact, complexity is a consideration beyond the practicalities of taking equilibrium model predictions to data. As Christos Papadimitriou put it: "We believe that this matter of computational complexity is one of central importance here, and indeed that the algorithmic point of view has much to contribute to the debate of economists about solution concepts. The reason is simple: If an equilibrium concept is not efficiently computable, much of its credibility as a prediction of the behavior of rational agents is lost – after all, there is no clear reason why a group of agents cannot be simulated by a machine. Efficient computability is an important modelling prerequisite for solution concepts." (Papadimitriou, 2007, pp. 29-30)

GS is a trading network with a set of agents F and set of contracts X (with choice functions that satisfy full substitutability and IRC) and an outcome A. The answer for an instance of GS is YES if the particular outcome A is not set-stable (that is, if there is a set of contracts Z that blocks A), otherwise the answer is NO.

Theorem 1. *Problem GS is NP-complete. Moreover, if choice functions are represented by oracles then finding the right answer for an instance of GS might need an exponential number of oracle calls.*

The non-existence of set-stable outcomes and their computational intractability motivates us to define a less restrictive stability notions.

We first define *trail stability*, which coincides with pairwise stability in a two-sided many-to-many matching market with contracts (Roth, 1984) and with chain stability in supply chains Ostrovsky (2008, p. 903). Define $T_f^{\leq m} = \{x_1, ..., x_m\} \cap T_f$ to be firm f's contracts out of the first m contracts in the trail and $T_f^{\geq m} = \{x_m, ..., x_M\} \cap T_f$ to be firm f's contracts out of the last $M - m + 1$ contracts in the trail.

Definition 7. An outcome $A \subseteq X$ is *trail-stable* if

1. A is acceptable.
2. There is no trail $T = \{x_1, x_2, \ldots, x_M\}$, such that $T \cap A = \emptyset$ and
   (a) x_1 is (A, f_1)-rational for $f_1 = s(x_1)$ and
   (b) At least one of the following two options holds:
      i. $T_{f_m}^{\leq m}$ is (A, f_m)-rational for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq M$, or
      ii. $T_{f_m}^{\geq m-1}$ is (A, f_m)-rational for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq M$
   (c) x_M is (A, f_{M+1})-rational for $f_{M+1} = b(x_M)$.

   The above trail T is called a *blocking trail to A*.

Trail stability is a natural stability concept when firms interact mainly with their buyers and suppliers and deviations by arbitrary sets of firms are difficult to arrange. In a trail-stable outcome, no agent wants to drop his contracts and there exists no set of *consecutive* bilateral contracts comprising a trail preferred by all the agents in the trail to the current outcome. First, f_1 makes an unilateral offer of x_1 (the first contract in the trail) to the buyer f_2. At this stage seller f_1 does not consider whether he may act as a buyer or a seller in the trail again (in that sense the deviations are pairwise and consecutive). The buyer f_2 then either unconditionally accepts the offer (forming a blocking trail) or conditionally

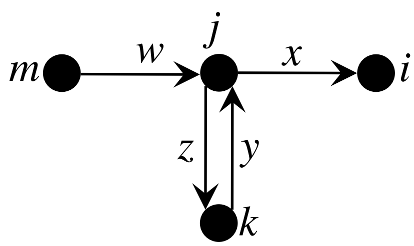

Figure 5: Example of a network that is trail-stable, but not set-stable

accepts the seller's offer while looking to offer a contract (x_2) to another buyer f_3. If f_2's buyer in x_2 happens to be f_1, then f_1 considers the offer of x_2 together with x_1 (which he has already offered). If f_1 accepts, we have a blocking trail. If f_2's buyer is not f_1, then his buyer either accepts x_2 unconditionally or looks for another seller f_4 after a conditional acceptance of x_2. The trail of "conditional" contracts continues until the last buyer f_{M+1} in the trail unconditionally accepts the upstream contract offer x_M.[11] Note that as the blocking trail grows, we ensure that each intermediate agent wants to choose all his contracts along the trail.

In general, without acyclicity, trail stability is a weaker stability notion than set stability.[12] The following example illustrates that trail-stable outcomes are not necessarily set-stable.[13]

Example 1 (Trail-stable outcomes are not necessarily set-stable)**.** Consider four contracts x, y, z and w. Assume that $i = b(x)$, $j = s(x) = s(z) = b(y) = b(w)$, $k = b(z) = s(y)$ and $m = s(w)$ (see Figure 5). Agents have the following preferences that induce fully substitutable choice functions:[14]

$\succ_i$: $\{x\} \succ_i \emptyset$

$\succ_m$: $\{w\} \succ_m \emptyset$

$\succ_j$: $\{x, y, w\} \succ_j \{z, y, w\} \succ_j \{x, y\} \succ_j \{z, y\} \succ_j \{w\} \succ_j \emptyset$

$\succ_k$: $\{z, y\} \succ_k \emptyset$.

Hence, a trail-stable outcome exists: $A = \{w\}$.The trail-stable outcome $\{w\}$ is Pareto inefficient as $\{z, y, w\}$ makes j and k better off without making i and m worse off. There is, however, no set-stable outcome.[15]

[11] The trail and the order of conditional acceptances can, of course, be reversed with f_{M+1} offering the first upstream contract to seller f_M and so on.

[12] See Lemma 5 below.

[13] This is similar to examples in Fleiner (2009, p. 12) and Hatfield and Kominers (2012, Fig. 3, p. 13).

[14] In all our examples, $\succ$ denotes a strict preference relation. Choice function induced by strict preferences satisfy IRC.

[15] Because $\{w\} \succ_j \{x, w\} \succ_k \{x, z, w\} \succ_{i,j} \{z, y, w\} \succ_{j,k} \{w\}$ and other outcomes are not acceptable.

To illustrate trail stability further, let us drop agents i and m and their corresponding contracts from the example above. The new preferences of j are $\{y, z\} \succ_j \emptyset$. There is one set-stable outcome $\{y, z\}$. There are, however, two trail-stable outcomes: $\emptyset$ and $\{y, z\}$. Is $\emptyset$ a reasonable possible outcome of this market? We argue that, in a variety of richer economic models of contracts, it may well be. Suppose that firms are unable to have a joint meeting and must resort to making a unilateral offers. Either firm may be reluctant to make the first offer because in absence of the counteroffer it could end up revealing sensitive information about its costs. Therefore, firms are unable to coordinate $\{y, z\}$ and are stuck in the "inefficient equilibrium". As such, trail stability provides a natural solution concept for matching markets in which firms have limited ability to coordinate their decisions in the contract network.

3 Existence and properties of stable outcomes

We can now state the first key result of this paper.

Theorem 2. *In any contract network X if choice functions of F are fully substitutable and satisfy IRC then there exists a trail-stable outcome $A \subseteq X$.*

This theorem establishes a positive existence result for stable outcomes in general contract networks: under the usual assumptions, trail-stable outcomes always exist.[16]

3.1 Fully trail-stable outcomes

In order to examine the structure of trail-stable outcomes, we need to introduce another stability notion.

Definition 8. An outcome $A \subseteq X$ is *fully trail-stable* if

1. A is acceptable.
2. There is no trail $T = \{x_1, x_2, \ldots, x_M\}$, such that $T \cap A = \emptyset$ and
    (a) $\{x_1\}$ is (A, f_1)-rational for $f_1 = s(x_1)$, and
    (b) $\{x_{m-1}, x_m\}$ is (A, f_m)-rational for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq M$ and
    (c) $\{x_M\}$ is (A, f_M)-rational for $f_{M+1} = b(x_M)$.

[16] Our results do not contradict Theorem 5 on the non-existence of set-stable outcomes in Hatfield and Kominers (2012) since Theorem 2 only considers the existence of trail-stable outcomes.

The above trail T is called a *locally blocking trail to* A.

Full trail stability may, at first glance, appear to be an unappealing stability concept. While it rules out (locally) blocking trails, it does not require, as trail stability, that agents accept all their contracts along such blocking trails. More formally, a locally blocking trail may not be an acceptable blocking trail. However, full trail stability has an interesting and important economic interpretation. Suppose contracts only need to be fulfilled sequentially i.e. once a firm's upstream contract has been fulfilled, it immediately fulfils its downstream contract.[17] This is a natural assumption in sequential production networks as production may not be able to continue without inputs and inputs would not be bought without a standing order. Then firms do not need to worry about being involved in multiple chains of contracts along the trail since they never need to be fulfilled together. As such full trail stability can be a useful stability concept in production networks in which production is sequential rather than (possibly) simultaneous. Full trail stability may be a better stability concept for a short-run prediction of network stability whereas trail stability is more suitable for the long run. It turns out that fully trail-stable outcomes also exist in general production networks.

Lemma 2. In any contract network X if choice functions of F are fully substitutable and satisfy IRC then there exists a fully trail-stable outcome $A \subseteq X$.

In order to prove Lemma 2, we use tools familiar to matching theory, such as the Tarski fixed-point theorem (Adachi, 2000, Fleiner, 2003, Echenique and Oviedo, 2006, Hatfield and Milgrom, 2005, Ostrovsky, 2008, Hatfield and Kominers, 2012). Let X^B and X^S be two subsets of X which represent the set of contracts offered to buyers and sellers. We define an isotone operator Φ that acts on (X^B, X^S) and any fixed-point $(\dot{X}^B, \dot{X}^S)$ of Φ corresponds to a fully trail-stable outcome $A = \dot{X}^B \cap \dot{X}^S$. These tools allow us to explore properties of fully trail-stable outcomes that have previously only been explored in a supply-chain or a two-sided setting. Recall that in the marriage model of Gale and Shapley, the existence of man-optimal and woman-optimal stable matchings follow from the well-known lattice structure of stable matchings. The key to extending this result to contract networks is to consider only terminal agents. We say that a fully trail-stable outcome A_{max} (A_{min}) that is *buyer-optimal* (*seller-optimal*) if any terminal buyer (terminal seller) prefers it to any other outcome i.e. for any fully trail-stable $Z \subseteq X$, we have that $C^f(A_{max} \cup Z) = A_{f,max}$.

Lemma 3. In any contract network X if choice functions of F are fully substitutable and satisfy IRC then the set of fully trail-stable outcomes contains buyer-optimal and seller-optimal outcomes.

[17] Alternatively, contracts further down the trail could be specified to be fulfilled later.

Lemma 3 extends Theorem 2 by Ostrovsky (2008) and Theorem 4 by Hatfield and Kominers (2012), which establish the existence of buyer- and seller-optimal outcomes in acyclic trading networks.[18] We say that $Y \subseteq X$ is *terminal-fully-trail-stable* if there is a fully trail-stable outcome $A \subseteq X$ such that $Y = A_{\mathcal{T}}$.

Lemma 4. In any contract network X if choice functions of F are fully substitutable and satisfy LAD/LAS then the terminal-fully-trail-stable contract sets form a lattice under terminal superiority.

Lemma 4 shows that whenever LAD/LAS holds choice functions of terminal agents define a natural partial order on outcomes and the terminal-fully-trail-stable contract sets form a lattice under this order. Note that for the lattice and the opposition-of-interests structure, only terminal agents play a role: two outcomes are equivalent if all the terminal agents have the same set of contracts. Indeed, if A^1 and A^2 are fully trail-stable outcomes then there is a fully trail-stable outcome A^+ such that all terminal buyers prefer A^+ to both A^1 and A^2 and all sellers prefer any of A^1 and A^2 to A^+.[19] This establishes full "polarization of interests" in trail-stable outcomes in the sense of (Roth, 1985) and immediately implies the existence of buyer-optimal (A_{max}) and seller-optimal (A_{min}) fully trail-stable outcomes. Therefore, our result substantially strengthens and generalizes the previous results by Roth (1985), Blair (1988), Echenique and Oviedo (2006) and Hatfield and Kominers (2012).[20]

The lattice structure of fully-trail stable outcomes allows us to straightforwardly extend two well-known properties of stable outcomes that have been known in two-sided matching markets and acyclic contract networks. One such property is the classic "rural hospitals theorem", which shows that in every stable allocation of a two-sided many-to-one doctor-hospital matching market, the same number of doctors are matched to every hospital (Roth, 1986). In buyer-seller networks, we can instead consider the difference between the number of upstream and downstream contracts that firms sign (Hatfield and Kominers, 2012). The following proposition gives the most general rural hospital theorem result.[21]

Proposition 1. Suppose that in a contract network X choice functions of F are fully substitutable and satisfy LAD/LAS. Then, for each firm, the difference between the number

[18]This is a common property of stable outcomes in two-sided markets with substitutable choice functions, however, it typically fails in richer matching models (Pycia and Yenmez, 2015, Alva, 2015a, Alva and Teytelboym, 2015).

[19]Of course, the same holds for if we exchange the role of buyers and sellers.

[20] Theorem 4 in Fleiner (2014), which states that any two stable flows agree on terminal contracts, is a further strengthening of Lemma 4 in the special case of network flows.

[21]Its proof, which we omit, follows the proof of Theorem 8 in Hatfield and Kominers (2012) word-for-word, only replacing "stable" with "fully trail-stable".

of upstream contracts and the number of downstream contracts is invariant across fully trail-stable allocations.

The lattice structure of fully trail-stable outcomes also gives a (somewhat weak) mechanism design result. A mechanism $\mathcal{M}$ is a mapping from a profile of agents' choice functions, $\mathbf{C}^F = (C^f)_{f \in F}$, to the set of outcomes.

Definition 9. A mechanism is *group strategy-proof* for $G \subseteq F$ if for any $\bar{G} \subseteq G$, there does not exist a choice function profile $\bar{\mathbf{C}}^{\bar{G}}$ such that for outcomes $\bar{A} = \mathcal{M}(\bar{\mathbf{C}}^{\bar{G}}, \mathbf{C}^{F \setminus \bar{G}})$ and $A = \mathcal{M}(\mathbf{C}^F)$ we have that $C^f(\bar{A} \cup A) = \bar{A}$ for every $f \in \bar{G}$.

A mechanism is group strategy-proof for a group of agents if they cannot jointly manipulate their choice functions and obtain an outcome that is better for all of them. Like Hatfield and Kominers (2012), we are only going to consider group strategy-proofness for terminal agents. We generalize their Theorem 10 with the following result.

Proposition 2. Suppose that in a contract network X choice functions of F are fully substitutable and satisfy LAD/LAS and additionally all terminal buyers (terminal sellers) demand at most one contract, then any mechanism that selects the buyer-optimal (seller-optimal) fully trail-stable allocation is group strategy-proof for all terminal buyers.

As is well known, the assumptions that underpin Proposition 2 – unit demands and extreme one-sidedness – cannot be substantially relaxed.[22]

The second set of properties of fully trail-stable outcomes concerns the effect of entry and exit of new firms in the trading network. This type of comparative static analysis is well-studied in two-sided matching markets (Gale and Sotomayor, 1985, Crawford, 1991, Blum et al., 1997, Hatfield and Milgrom, 2005). More recently, Ostrovsky (2008) and Hatfield and Kominers (2013) extended these results the case of supply chains.

First, let us consider what happens when a terminal seller is added to the market. More formally, let $F' = F \cup f'$ and let A'_{max} and A'_{min} be the buyer-optimal and the seller-optimal fully trail-stable outcomes in F' respectively.

Proposition 3. Consider a contract network X in which choice functions of F are fully substitutable and satisfy IRC. Suppose a new terminal seller f' whose choice function is fully substitutable and satisfies IRC enters the market.

Then every terminal seller $f \neq f'$ is prefers A_{max} to A'_{max} and prefers A_{min} to A'_{min}, and each terminal buyer f is prefers A_{max} to A'_{max} and prefers A_{min} to A'_{min}.
The opposite holds when f' is terminal buyer.

[22]Its proof, which we omit, just follows the proof of Theorem 1 in Hatfield and Kojima (2009) (which was pointed out by Hatfield and Kominers (2012)).

The proposition says that with a new seller, the seller-optimal outcome A_{min} and the buyer-optimal outcome A_{max} move in the direction favorable to terminal buyers and unfavorable to terminal sellers. Symmetrically, when a terminal buyer is added or if a seller leaves, A_{min} and A_{max} move in the opposite direction. In other words, more competition on one end of an industry is bad for the agents on that end and good for the agents on the other end. This proposition generalizes Theorem 3 in Ostrovsky (2008).

Now consider the following *market readjustment process*: When the new terminal seller f' enters, and we already have a fully trail-stable outcome A with corresponding fixed point $(\dot{X}^B, \dot{X}^S)$ then let X be the set of all contracts in the new network, and let us define $(\dot{X}'^B, \dot{X}'^S) = (\dot{X}^B, \dot{X}^S \cup X_{f'})$. Operator Φ' acts on (X'^B, X'^S) using choice functions of F'. Let $(\hat{X}^B, \hat{X}^S)$ be the fixed point of the iteration of fuction Φ, with associated outcome $\hat{A} = \hat{X}^B \cap \hat{X}^S$. This $\hat{A}$ be the result of the market readjustment process.

Proposition 4. Consider a contract network X in which choice functions of F are fully substitutable and satisfy IRC and A is a fully trail-stable outcome with associated buyer and seller offer sets X^B and X^S. Suppose a new terminal seller f' whose choice function is fully substitutable and satisfies IRC enters the market and let $\hat{A}$ be the result of the market readjustment process.

Then, all terminal sellers prefer A to $\hat{A}$ and all terminal buyers (other than f') prefer $\hat{A}$ to A.
The opposite holds when f' is terminal buyer.

An analogous result can be obtained when terminal buyers and terminal sellers exit the market so this proposition generalizes the Theorem in Hatfield and Kominers (2013).

3.2 Relationships between stability concepts

In this section, we link together all the stability concepts discussed above. We first show that set stability implies full trail stability, which in turn implies trail stability. We also link set stability and trail stability via intensity. We then link trail stable and fully trail-stable outcome via separability. Finally, we explore path stability (Ostrovsky, 2008).
The follow lemma ties three key stability concepts together.

Lemma 5. In any contract network X if choice functions of F are fully substitutable and satisfy IRC then the following hold for an outcome $A \subseteq X$:

(i) If A is a fully trail-stable outcome then A is also trail-stable.

(ii) If A is a set-stable outcome then A is fully trail-stable.

Lemma 2 and Lemma 5 immediately imply Theorem 2. We now pin down the role of separability for trail-stable and fully trail-stable outcomes.

Proposition 5. In any contract network X whenever choice functions of F are fully substitutable, separable and satisfy IRC an outcome $A \subseteq X$ is fully trail-stable if and only if it is trail-stable.

Under separability all properties of fully trail-stable outcomes apply to trail-stable outcomes. This is summarized in the following corollary.[23]

Corollary 1. *Suppose that in a contract network X, choice functions of F are fully substitutable, separable and satisfy IRC. Then all properties of fully trail-stable outcomes, described in Lemma 3, Lemma 4, Proposition 1, Proposition 2, Proposition 3 and Proposition 4, apply to trail-stable outcomes.*

Separability is crucial for the correspondence between fully trail-stable and trail-stable outcomes. Separability ensures that all blocking trails are locally blocking trails. An example below shows that full trail stability is strictly stronger that trail stability.

Example 2 (Trail-stable outcomes are not always fully trail-stable)**.** Consider agents and contracts described in Example 1 and Figure 5. Agents have the following preferences that induce fully substitutable choice functions:
$\succ_m$: $\{w\} \succ_m \emptyset$
$\succ_i$: $\{x\} \succ_i \emptyset$
$\succ_k$: $\{z, y\} \succ_k \emptyset$
$\succ_j$: $\{z, y\} \succ_j \{w, z\} \succ_j \{y, x\} \succ_j \emptyset$.
The empty set is preferred to any other set of contracts.
For outcome $A = \emptyset$, the trail $\{w, z, y, x\}$ is locally blocking trail but not trail-blocking. Therefore, trail-stable outcomes are $\emptyset$ and $\{z, y\}$ but the only fully trail-stable outcome is $\{z, y\}$. Note that j's choice function induced by the preference is not separable.

Finally, we explore the relationship between trail stability, set stability and path stability.

Definition 10. Choice functions of $f \in F$ are *simple* if there exists an "intensity" mapping $w : X_f \to \mathbb{R}$ such that whenever A is a (W, f)-rational set for some acceptable set A of contracts, then for every $y \in X_f^B \cap A$ there exists $z \in X_f^S \cap A$ such that $w(y) > w(z)$ holds.

[23] We also conjecture that in any contract network X if choice functions of F satisfy full substitutability and *only* LAD/LAS then the terminal-trail-stable contract sets form a lattice under terminal superiority, but leave this for future work.

One example of choice functions which are simple are the following: if the agent is offered a set of contracts, he picks the upstream contract y with the highest intensity and a downstream contract z with the lowest intensity (as long as the intensity of the y is greater than of z, otherwise he picks nothing). For example, if the intensity mapping w represents the per-unit price of the contract, then the condition says that the firm only signs a pair of contracts if the price in the downstream contract is greater than the price in the upstream contract, while picking the highest-price downstream contract and the lowest-price upstream contract.

Proposition 6. In any contract network X whenever choice functions of F are fully substitutable, simple and satisfy IRC then an outcome $A \subseteq X$ is set-stable if and only if it is trail-stable.

We now formally define path stability, introduced by Ostrovsky (2008) in the context of acyclic networks as "chain stability".[24] To recap, a path C is a trail in which all the agents are distinct. Path-stable outcomes rule out consecutive pairwise deviations along path. While this stability concept was introduced in the context of acyclic trading networks, it could also be applicable to general trading networks in which firms only have one opportunity to recontract during a deviation.

Definition 11. An outcome $A \subseteq X$ is *path-stable* if

1. A is acceptable.
2. There is no path $C = \{x_1, x_2, \ldots, x_M\}$, such as $C \cap A = \emptyset$ and
   (a) $\{x_1\}$ is (A, f_1)-rational for $f_1 = s(x_1)$, and
   (b) $\{x_{m-1}, x_m\}$ is (A, f_m)-rational for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq M$ and
   (c) $\{x_M\}$ is (A, f_M)-rational for $f_{M+1} = b(x_M)$.

Since every path is a trail, every trail-stable outcome is path-stable. In acyclic networks every trail is also path, so path-stable, trail-stable and fully trail-stable outcomes coincide with set-stable outcomes (Hatfield and Kominers, 2012). However, as the example below shows, path stability is weaker than trail stability (and hence weaker than full trail stability) in general contract networks.

[24] As we mentioned before, we also want to distinguish it from "chain stability" introduced by Hatfield et al. (2015) for general networks, which we refer to as "strong-trail stability".

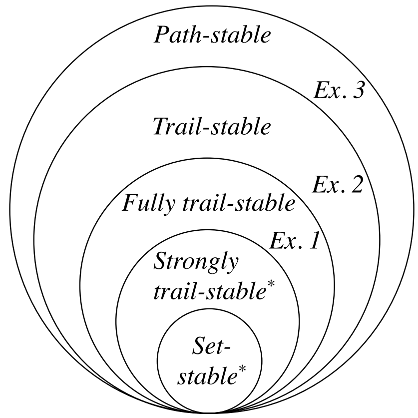

Figure 6: Relationship between stability concepts in general contract networks

Example 3 (Path-stable outcomes are not necessarily trail-stable)**.** Consider agents and contracts described in Examples 1 and 2, and Figure 5. Agents have the following fully substitutable preferences:

$\succ_m$: $\{w\} \succ_m \emptyset$
$\succ_i$: $\{x\} \succ_i \emptyset$
$\succ_k$: $\{z, y\} \succ_k \emptyset$
$\succ_j$: $\{w, x, z, y\} \succ_j \{w, z\} \succ_j \{y, x\} \succ_j \{y, z\} \succ_j \emptyset$
The empty set is preferred to any other set of contracts.
Now, for outcome $\emptyset$, the trail $\{w, z, y, x\}$ is trail-blocking, but there is no blocking path for $A = \emptyset$. Outcome $\{z, y\}$ is, however, blocked by path $\{w, x\}$. Therefore the only trail-stable outcome is $\{w, z, y, x\}$ and the path-stable outcomes are $\emptyset$ and $\{w, z, y, x\}$.

This is intuitive because paths allows the firms to appear in the blocking set only once therefore they rule out fewer possible blocks. Figure 6 summarizes the relationships between various stability concepts in general contract networks. Set-stable and strongly trail-stable outcomes may not exist and they are only equivalent under the Laws of Aggregate Demand and Supply as Example 1 in Hatfield et al. (2015) shows.

4 Competitive equilibrium

We now turn to the existence of competitive equilibrium in our model when each contract specifies a price. We assume that each contract $(\omega, p_\omega) := x \in X := \Omega \times \mathbb{Z}$ specifies a trade $\omega \in \Omega$ and a price $p_\omega \in \mathbb{Z}$.[25] Trades from $\Psi \subseteq \Omega$ involving f are denoted Ψ_f. For any p_ω,

[25] This price can be viewed as a *generalized salary*; see the discussion by Roth (1984). Since firms can sign more than one contract between them, our framework with contracts cannot be embedded into a framework

we have $b(\omega) = b(x)$ and $s(\omega) = s(x)$. Trades associated with contracts Y are denoted $\tau(Y)$. Let p be $|\Omega|$-dimensional price vector specifying a price for each trade. A set of contracts $Y \subseteq X$ is feasible if there is at most one price specified for each trade i.e. there is no trade ω and price p_ω and $p'_\omega \neq p_\omega$ such that (ω, p_ω) and (ω, p'_ω) are in Y. Choice functions are *feasible* if for any $Y \subseteq X$, $C^f(Y)$ is feasible. An arrangement $[\Psi; p]$ is a set of trades $\Psi \subseteq \Omega$ and a price vector specifying precisely one price for each trade in the economy. Call $\kappa([\Psi; p]) = \cup_{\omega \in \Psi}(\omega, p_\omega)$ the set of contracts associated with the arrangement $[\Psi; p]$. Clearly, $\kappa([\Psi; p])$ is feasible. Competitive equilibrium specifies the allocation of trades and the prices of every trade in the economy.

Definition 12. Competitive equilibrium is an arrangement $[\Psi^*; p^*]$ such that for all $f \in F$, $\kappa([\Psi^*_f, p^*]) = C^f(\kappa([\Omega, p^*]))$.

We can construct a competitive equilibrium outcome (i.e. a feasible set of contracts) from any competitive arrangement by associating the contracts with the realized trades at competitive equilibrium prices. In order to ensure that prices are indeed assigned to every trade, we introduce two further assumptions:

Definition 13. *Complete prices* (CP): For every $\omega \in \Omega$:

1. There exists a price $\hat{p}_\omega$ such that whenever $f = b(\omega)$, $(\omega, \hat{p}) \in C^f((\omega, \hat{p}) \cup Y)$ for any $Y \subseteq X$.

2. There exists a price $\check{p}_\omega$ such that whenever $f = s(\omega)$, $(\omega, \check{p}) \in C^f((\omega, \check{p}) \cup Y)$ for any $Y \subseteq X$.

3. Whenever $(\omega, p_\omega) \in R^{s(\omega)}((\omega, p_\omega) \cup Y)$ and $(\omega, p_\omega) \in R^{b(\omega)}((\omega, p_\omega + 1) \cup Y)$, there exists a price $p_\omega \leq \tilde{p}_\omega \leq p_\omega + 1$, such that $(\omega, \tilde{p}) \in R^{b(\omega)}((\omega, \tilde{p}_\omega) \cup Y), R^{s(\omega)}((\omega, \tilde{p}_\omega) \cup Y)$.

(CP1) says that there exists a vector of (low) prices at which firms want to buy all their upstream trades; (CP2) says that there exists a vector of (high) prices at which firms want to sell all their downstream trades; (CP3) says whenever a seller rejects a contract for a trade at a particular price and the buyer rejects a contract for the same trade at a higher price, there exists a price (either p_ω or $p_\omega + 1$) for the trade at which they both reject the contract whenever the set of other offered contracts is unchanged. It is worth highlighting that (CP3) would be innocuous if prices were continuous.

Definition 14. *Price Monotonicity* (PM): Consider an outcome A and two other contracts $\{x, x'\} \notin A$ for the same trade ω that differ only in price i.e. $x = (\omega, p_\omega)$ and $x' = (\omega, p'_\omega)$

with prices despite the full substitutability assumption (Echenique, 2012, Kominers, 2012).

such that $p_\omega > p'_\omega$. If $f = b(x) = b(x')$, then $x \notin C^f(A \cup \{x, x'\})$, and if $f = s(x) = s(x')$, then $x' \notin C^f(A \cup \{x, x'\})$.

This assumption says that all things being equal firms strictly prefer to buy a cheaper upstream trade and to sell a more expensive downstream trade. It extends the "generalized salary condition" (Roth, 1985) or "Pareto separability" (Roth, 1984) used in the context of two-sided markets in contract networks.

Our price-adjustment process mimics the one described by Kelso and Crawford (1982) and Roth (1984) except that now the sellers are not bound by the agreed contracts.[26] In fact, it is a special case of operator Φ applied on a set of contracts used to prove Lemma 2 but since we are able to keep track of the prices of all trades and we can find supporting competitive equilibrium prices once the process terminates and finds a (fully) trail-stable allocation. The intuition here is that prices of over-demanded trades increase. Initially, every upstream trade is demanded by the buyers. Buyers continue to raise prices of (upstream) trades until every demanded (upstream) trade has a supplier or we can find a set of prices at which neither party wants to trade. This is a (fully) trail-stable contract allocation and from here we can construct competitive equilibrium prices to support it.[27] This gives us our final result.

Theorem 3. *Consider a set of contracts X that specifies trades and prices and assume that choice functions of F are fully substitutable, feasible and satisfy IRC. In addition, assume that (CP) and (PM) are satisfied. Then a competitive equilibrium arrangement exists and a competitive equilibrium outcome is (fully) trail-stable.*

5 Conclusion

Set stability is an appealing stability concept, but in general contract networks set-stable outcomes may not exist and they are not computationally tractable. In this paper, we introduced a new natural stability notion for general contract networks, called trail stability. We showed that any contract network has a trail-stable outcome when choice functions are fully substitutable. We then showed that outcomes satisfying an even stronger stability concept – full trail stability – always exist and have a natural lattice structure and inherit a host of properties studied in two-sided and supply-chain settings. Moreover, we described how set-stable outcomes, path-stable and (fully) trail-stable outcomes are related in general networks. We then showed that in networked markets competitive equilibrium can exist

[26]The price-adjustment process is analogous with buyers not being tied to contracts.

[27]Hatfield et al. (2013, Theorem 6) show that supporting competitive equilibrium prices can also be found for any stable contract allocation when utility functions are quasilinear. Their proof is rather different.

without the quasilinear assumption on utility functions. Full substitutability is crucial for existence of trail-stable outcomes since previous maximal domain results for many-to-many matching markets apply in our case (see, for example, Hatfield and Kominers (2012, Theorem 6) and Hatfield and Kominers (2015a, Theorem 2)). When firms have quasilinear utility functions, (full) substitutability is not necessary for competitive equilibrium and even when all agents have complementary preferences competitive equilibrium may exist (Baldwin and Klemperer, 2013, Drexl, 2013, Hatfield and Kominers, 2015b, Teytelboym, 2014). Although Alva and Teytelboym (2015) show that trail-stable outcomes exist in supply chains even in the presence of upstream complementarities with general choice functions, it is not clear whether this result can be extended to general contract networks. Finally, it would be interesting to understand how this model can be tested and estimated empirically. This is a fruitful area for future research.

A Appendix

Proof of Lemma 1. Consider $Y \subseteq X_f$ and $z \in X_f^B \setminus Y$ such that $z \notin C^f(Y \cup \{z\})$. Then, from SSS, $C_B^f(Y \cup \{z\}) \subseteq C_B^f(Y)$ and from CSC $C_S^f(Y \cup \{z\}) \supseteq C_S^f(Y)$. If choice functions satisfy LAD/LAS then $|C_B^f(Y)| - |C_S^f(Y))| \leq |C_B^f(Y \cup \{z\})| - |C_S^f(Y \cup \{z\})|$ so there must be equality, so $C^f(Y \cup \{z\}) = C^f(Y)$. □

A.1 Proof of Theorem 1

Proof of Theorem 1. Problem GS is clearly belongs to complexity class NP as verifying that Z is a blocking set for A requires a polynomial time proof.

To show that GS is NP-hard we reduce the NP-complete partition problem to GS. An instance of the partition problem is given by a k-tuple $A = (a_1, a_2, \ldots, a_k)$ of positive integers such that $a_1 \leq a_2 \leq \ldots \leq a_k$ holds. The answer to this problem is YES if and only if there is a subset I of $\{1, 2, \ldots, k\}$ such that $\sum_{i \in I} a_i = s$ where $2s = \sum_{i=1}^k a_i$. So assume that the partition problem is given by $\mathcal{A} = (a_1, a_2 \ldots a_k)$. Construct a trading network with firms f and g and with contracts y and x_i such that $f = s(y) = b(x_i)$ and $g = b(y) = s(x_i)$ for $i \in \{1, 2, \ldots, k\}$. Define choice function $C_{\mathcal{A}}^f$ with the help of $s := \frac{1}{2}\sum_{i=1}^k a_i$ by

$$C_{\mathcal{A}}^f(X|Y) = \begin{cases} (X|Y) & \text{if } \sum\{a_i : x_i \in X\} \geq s \\ (X|\emptyset) & \text{if } \sum\{a_i : x_i \in X\} < s \end{cases}$$

It is easy to check that $C_{\mathcal{A}}^f$ satisfies the full substitutability condition and IRC. Define $C_{\mathcal{A}}^g$ as follows:

$$C_{\mathcal{A}}^g(Y|X) = \begin{cases} (\emptyset|\emptyset) & \text{if } Y = \emptyset \\ (Y|X) & \text{if } Y = \{y\} \text{ and } \sum\{a_i : x_i \in X\} \leq s \\ (Y|X \cap \{x_1, x_2, \ldots, x_t\}) & \text{if } Y = \{y\} \text{ and } \sum\{a_i : x_i \in X, i \leq t\} \leq s < \sum\{a_i : x_i \in X, i < t+1\} \end{cases}$$

One can readily check that $C_{\mathcal{A}}^g$ also satisfies the full substitutability condition and IRC. That is, based on the partition problem instance, we have determined a trading network. To define our GS instance, define an outcome $A = \emptyset$. We have to show that the answer to the partition problem is YES if and only if $A = \emptyset$ is not set-stable.

Assume now that the answer to our partition problem instance is YES, that is $\sum_{i \in I} a_i = s$. Define $X_I := \{x_i : i \in I\}$ and $Y = \{y\}$. By the above definitions, $C_{\mathcal{A}}^f(X|Y) = (X|Y)$ and $C_{\mathcal{A}}^g(Y|X) = (Y|X)$, hence $X \cup Y$ blocks $A = \emptyset$, so A is not set-stable.

Assume now that $A = \emptyset$ is not set-stable. This means that there is a blocking set Z to A and define $I = \{i : x_i \in Z\}$, $X_I := \{x_i : x_i \in Z\}$ and $Y := Z \cap \{y\}$. As Z is blocking, we have $C_{\mathcal{A}}^f(X_I|Y) = (X_I|Y)$ and $C_{\mathcal{A}}^g(Y|X_I) = (Y|X_I)$. If $Y = \emptyset$ then $(Y|X_I) = C_{\mathcal{A}}^g(Y|X_I) = C_{\mathcal{A}}^g(\emptyset|X_I) = (\emptyset, \emptyset)$, so $Z = X_I \cup Y = \emptyset \cup \emptyset = \emptyset$, and hence Z is not blocking. Otherwise, $Y = \{y\}$, and from $C_{\mathcal{A}}^g(Y|X_I) = (Y|X_I)$ we get that $\sum_{i \in I} a_i \leq s$. Moreover, from $y \in C_{\mathcal{A}}^f(X_I, Y)$ we get that $\sum_{i \in I} a_i \geq s$. Consequently $\sum_{i \in I} a_i = s$, and the answer to the partition problem is YES.

To prove the second part of the theorem, define a contract network with firms f and g and with contracts y and x_i such that $f = s(y) = b(x_i)$ and $g = b(y) = s(x_i)$ for for

$1 \leq i \leq 2n$. Define the following choice function

$$C_0^f(X|Y) = \begin{cases} (X|Y) & \text{if } |X| \geq n+1 \\ (X|\emptyset) & \text{if } |X| \leq n \end{cases} \tag{A.1}$$

For $I \subseteq \{1, 2, \ldots, n\}$ define $X_I := \{x_i : i \in I\}$. For $|I| = n$ let

$$C_I^f(X|Y) = \begin{cases} (X|Y) & \text{if } |X| \geq n+1 \text{ or if } X = X_I \\ (X|\emptyset) & \text{if } |X| \leq n \text{ and } X \neq X_I \end{cases}$$

It is straightforward to check that choice functions C_0^f and C_I^f above satisfy the full substitutability condition and IRC. Define the following choice function for g

$$C^g(Y|X) = \begin{cases} (\emptyset|\emptyset) & \text{if } Y = \emptyset \\ (Y|X) & \text{if } Y = \{y\} \text{ and } |X| \leq n \\ (Y|X \cap \{x_1, x_2, \ldots, x_t\}) & \text{if } Y = \{y\} \text{ and } |\{x_i \in X : i \leq t\}| = n \end{cases} \tag{A.2}$$

As $C^g = C^g_{\mathcal{A}}$ for $\mathcal{A} = (1, 1, \ldots, 1)$, C^g also satisfies the full substitutability condition and IRC.

Now assume that an instance of problem GS is given by the above network and an outcome $A = \emptyset$. Assume that the choice functions are not given explicitly, but by value-returning oracles. Moreover, we know exactly that the choice function of g is the one defined in (A.2) and we know that the choice function of f is either C_0^f or C_I^f for some I. It is easy to check that A is not set-stable if and only if $C^f = C_I^f$ and in this case the only blocking set is $Z = X_i \cup \{y\}$. So if one has to decide set stability of A, then one must determine the $C^f(Z)$ values for all such possible Z, and this means $\binom{2n}{n}$ oracle calls. □

A.2 Proof of Lemma 2

Consider Y^B and Z^S, which are subsets of X, and represent sets of available upstream and downstream contracts for all agents, respectively. Define a lattice L with the ground set $X \times X$ with an order $\sqsubseteq$ such that $(Y^B, Z^S) \sqsubseteq (Y'^B, Z'^S)$ if $Y^B \subseteq Y'^B$ and $Z^S \supseteq Z'^S$.

Furthermore, define a mapping Φ as follows:

$$\begin{aligned} \Phi_B(Y^B, Z^S) &= X \setminus R_S(Z^S|Y^B) \\ \Phi_S(Y^B, Z^S) &= X \setminus R_B(Y^B|Z^S) \\ \Phi(Y^B, Z^S) &= (\Phi_B(Y^B, Z^S), \Phi_S(Y^B, Z^S)) \end{aligned}$$

where $R_S(Z^S|Y^B) := \bigcup_{f \in F} R_S^f(Z^S|Y^B)$ and $R_B(Y^B|Z^S) := \bigcup_{f \in F} R_B^f(Y^B|Z^S)$. Clearly, Φ is isotone (Fleiner, 2003, Ostrovsky, 2008, Hatfield and Kominers, 2012) on L. We rely on the following well-known fixed point theorem of Tarski.

Theorem A.1. *(Tarski, 1955) Let L be a complete lattice and let $\Phi : L \to L$ be an isotone mapping. Then the set of fixed points of Φ in L is also a complete lattice.*

Proof of Lemma 2. Existence of fixed-points of Φ follows from Theorem A.1 since $(X \times X, \sqsubseteq)$

is a complete lattice.[28]

We claim that every fixed point $(\dot{X}^B, \dot{X}^S)$ of Φ corresponds to an outcome $\dot{X}^B \cap \dot{X}^S = A$ that is fully trail-stable. First, we show that A is individually rational. Observe that if $(\dot{X}^B, \dot{X}^S)$ is a fixed point then $\dot{X}^S \cup \dot{X}^B = X$. To see this suppose for contradiction that there is a contract $x \notin \dot{X}^S \cup \dot{X}^B$. Then $x \notin R_S(\dot{X}^S|\dot{X}^B)$ therefore $x \in X \setminus R_S(\dot{X}^S|\dot{X}^B) = \dot{X}^B$. So x is has to be in $\dot{X}^S \cup \dot{X}^B$ This implies that $R_S(\dot{X}^S|\dot{X}^B) = X \setminus \dot{X}^B = \dot{X}^S \setminus A$ so $C_S(\dot{X}^S|\dot{X}^B) = A$ and similarly $C_B(\dot{X}^B|\dot{X}^S) = A$ From this, we can see that A is individually rational.

Second, we show that A is fully trail-stable. This is similar to Step 1 of the Proof of Lemma 1 in Ostrovsky (2008). Suppose that $T = \{x_1, ...x_m\}$ is a locally blocking trail and assume towards a contradiction that $T \cap A = \emptyset$. Since we have that $x_1 \in C_S^{s(x_1)}(A \cup x_1|A)$, we must have that $x_1 \in C_S^{s(x_1)}(\dot{X}^S \cup x_1|A)$ Since if $C_S^{s(x_1)}(\dot{X}^S \cup x_1|A) \subseteq \dot{X}^S$ then by IRC $C_S^{s(x_1)}(\dot{X}^S \cup x_1|A) = A$, therefore $C_S^{s(x_1)}(A \cup x_1|A) = A$ Also, $x_1 \in C_S^{s(x_1)}(\dot{X}^S \cup x_1|\dot{X}^B)$ (by CSC). If $x_1 \in \dot{X}^S$, then $x_1 \in \dot{X}^B = X \setminus R_S(\dot{X}^S|\dot{X}^B)$. But we assumed that $x_1 \notin A$, so $x_1 \in \dot{X}^B$.

Now, consider x_2. By definition of a locally blocking trail, we have that $x_2 \in C_S^{s(x_2)}(A \cup x_2|A \cup x_1)$. Once again by full substitutability and IRC, we obtain that and $x_2 \in C_S^{s(x_2)}(\dot{X}^S \cup x_2|\dot{X}^B \cup x_1)$. If $x_2 \in \dot{X}^S$, then $x_2 \in \dot{X}^B = X \setminus R_S(\dot{X}^S|\dot{X}^B)$. But we assumed that $x_2 \notin A$, so $x_2 \in \dot{X}^B$. Now proceed by induction, we show that every $x \in T$ is in $\dot{X}^B$. Consider the last contract x_m. Since $x_m \in C_B^{b(x_m)}(A \cup x_m|A)$, using the same argument we had for x_1, we get that $x_m \in \dot{X}^S$. A contradiction.

Now we show that every fully trail-stable outcome corresponds to a fixed point:

Suppose A is fully trail-stable. For every $x_i \notin A$, if there exists a trail $\{x_1 x_2 \dots x_i\}$ such that x_1 is $(A, s(x_1))$-rational, and $\{x_{m-1}, x_m\}$ is (A, f_m)-rational for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq i$, then let $x_i \in X_0^B$, otherwise $x_i \in X_0^S$. Let $\dot{X}^B = A \cup X_0^B$ and $\dot{X}^S = A \cup X_0^S$. Clearly $\dot{X}^S \cup \dot{X}^B = X$.

Outcome A is individually rational, so $C^f(A) = A_f$ for all $f \in F$. For every firm f, if $f = s(x)$ and $x \in \dot{X}^S \setminus A$ then $x \notin C^f(A \cup \{x\})$ otherwise x would be in X_B. From SSS, $C_S^f(\dot{X}^S|A) = A$. And if $f = b(y)$ and $y \in \dot{X}^B \setminus A$ then $y \notin C^f(A \cup \{y\})$ otherwise the trail ending in y would be a locally blocking trail. From SSS, $C_B^f(\dot{X}^B|A) = A$. Moreover, $\{x, y\} \nsubseteq C(A \cup \{x, y\})$ otherwise x would be in $\dot{X}^B$. These together imply that $C_S(\dot{X}^S|\dot{X}^B) = A$ and $C_B(\dot{X}^B|\dot{X}^S) = A$. Therefore $R_S(\dot{X}^S|\dot{X}^B) = \dot{X}^S \setminus A$, $R_B(\dot{X}^B|\dot{X}^S) = \dot{X}^B \setminus A$, so $X \setminus R_S(\dot{X}^S|\dot{X}^B) = \dot{X}^B$, $X \setminus R_B(\dot{X}^B|\dot{X}^S) = \dot{X}^S$. □

A.3 Proof of Propositions 5, 6 and Lemma 5

The key to this is the following two useful lemmata. In the proofs, we will use the concept of a circuit, which is a closed trail.

Definition A.1. A non-empty sequence of different contracts $Q = \{x_1, \dots, x_M\}$ is a *circuit* if $b(x_m) = s(x_{m+1})$ holds for all $m = 1, \dots, M-1$, and $b(x_M) = s(x_1)$.

Lemma A.1. Let F be the set of agents and X be the set of contracts in a contract network with fully substitutable choice functions that satisfy IRC. If Y and Z are disjoint sets of

[28] Hence, we do not actually require the assumption of the finiteness of contracts as long as lattice L is appropriately defined. However, we maintain this assumption for ease of comparison with previous results.

contracts and f is an agent such that Z_f is (Y,f)-rational then for any contract z of Z_f^B one of the following options hold: (1) z is (Y,f)-rational or (2) there exists some $z' \in Z_f^S$ such that $\{z,z'\}$ is a (Y,f)-rational pair or (3) there are $z_1,z_2,\ldots,z_k \in Z_f^S$ such that both $\{z,z_1,z_2,\ldots,z_k\}$ and $\{z_i\}$ (for $1 \le i \le k$) are (Y,f)-rational. For $z \in Z_f^S$ an analogous statement holds.

Proof of Lemma A.1. We can suppose without loss of generality that $z \in X_f^B$. From the SSS property, it follows that $z \in C^f(Y_f \cup Z_f^S \cup \{z\})$.

Assume that $C^f(Y_f \cup Z_f^S \cup \{z\}) \cap Z_f^S = \emptyset$. Therefore, we have $C^f(Y_f \cup Z_f^S \cup \{z\}) \subseteq (Y_f \cup \{z\}) \subseteq (Y_f \cup Z_f^S \cup \{z\})$, so from IRC $z \in C^f(Y_f \cup \{z\})$, so z is (Y,f)-rational, we get option (1).

So if z is not (Y,f)-rational then there are some contracts $\{z_1,z_2\ldots z_k\} = C^f(Y_f \cup Z_f^S \cup \{z\}) \cap Z_f^S$. If there exists an z_i such that is z_i is not (Y,f)-rational, then using SSS again, we have $z_i \in C^f(Y_f \cup \{z,z_i\})$.

Suppose $z \notin C^f(Y_f \cup \{z,z_i\})$, then $C^f(Y_f \cup \{z,z_i\}) \subseteq (Y_f \cup \{z_i\})$, and from IRC we have $C^f(Y_f \cup \{z,z_i\}) = C^f(Y_f \cup \{z_i\})$. But since z_i is not (Y,f)-rational this is impossible, therefore $\{z,z_i\} \subseteq C^f(Y_f \cup \{z,z_i\})$, we achieved a (Y,f)-rational pair.

If all of $\{z_1,z_2\ldots z_k\}$ are (Y,f)-rational, we get option (3). □

A consequence of Lemma A.1 is that full trail stability is a stronger property than trail stability.

Later we are going to need the following lemma.

Lemma A.2. Let F be the set of agents and f be an agent in a contract network with fully substitutable choice functions that satisfy IRC. Assume that Y is acceptable and $x_1,x_2,\ldots,x_k \in X_f^B$ and $z_1,z_2,\ldots,z_k \in X_f^S$ such that $\{x_i,z_i\}$ is a (Y,f)-rational pair for any $1 \le i \le k$ but $\{x_1,x_2,\ldots,x_k,z_1,z_2,\ldots,z_k\}$ is not (Y,f)-rational. Then $\{x_i,z_j\}$ is a (Y,f)-rational pair for some $i \neq j$.

The above statement remains true if one or both of the contracts x_1 and z_k are *void*. When we say that x_1 is void, we mean that:

- x_1 is empty, so
- the trail starts with z_1, and
- instead of (Y,f)-rationality of pair $\{x_1,z_1\}$ we need (Y,f)-rationality of z_1 and $i \neq 1$ in the conclusion. When we say that both x_1 and z_k are void, we mean that there is (Y,f)-rational pair $\{x_i,z_j\}$, $i \neq j$ such that $\{x_i,z_j\} \neq \{x_k,z_1\}$.

Proof of Lemma A.2. Suppose, for example, that $z_j \notin C^f(Y \cup \{x_1,x_2,\ldots,x_k,z_1,z_2,\ldots,z_k\})$ for some j such that both x_j and z_j exist. Then from CSC, $z_j \notin C^f(Y \cup \{x_j,z_1,z_2,\ldots,z_k\})$. But $x_j \in C^f(Y \cup \{x_j z_j\})$ so from CSC $x_j \in C^f(Y \cup \{x_j,z_1,z_2,\ldots,z_k\})$. Since x_j is not (Y,f)-rational, there is a $z_l \in C^f(Y \cup \{x_j,z_1,z_2,\ldots,z_k\})$ therefore $\{x_j,z_l\}$ is (Y,f)-rational and $l \neq j$.

In the case that x_1 is void and $z_1 \notin C^f(Y \cup \{x_2,\ldots,x_k,z_1,z_2,\ldots,z_k\})$, from CSC, $z_1 \notin C^f(Y \cup \{z_1,z_2,\ldots,z_k\})$. This is impossible when z_1 is (Y,f)-rational but none of the other z_j contracts are (Y,f)-rational. Therefore if we have found (Y,f)-rational pair $\{x_i,z_j\}$, then at least one of x_i and z_j was not (Y,f)-rational by itself. □

Proof of Lemma 5. Consider a fully trail-stable outcome A. Suppose that A is not trail-stable, i.e. there exists a blocking trail $T = \{x_1, x_2 \dots x_M\}$ for it. Without the loss of generality, we may assume that (b)ii holds in Definition 7. The other case when (b)i holds can be proved analogously.

We are going to find indices $1 \leq i_1 < i_2 < i_3 \dots i_l \leq k$ such that

- x_{i_1} is $(A, s(i_1))$-rational, and
- $b(x_{i_{m-1}}) = s(x_{i_m}) = f_m$ and $\{x_{i_{m-1}}, x_{i_m}\}$ is a (A, f_m)-rational pair for all $1 < m \leq l$, and
- x_{i_l} is $(A, b(i_l))$-rational.

So this subset of the trail forms a locally blocking trail T'.

In the blocking trail T, choose the last contract $x_i \in T$ such that x_i is $(A, s(x_i))$-rational. There is at least one contract like this, since x_1 is $(A, s(x_1))$-rational by definition. Let $i_1 := i$.

Suppose we have already found $i_1 \dots i_m$ that satisfies our requirements. If x_{i_m} is $(A, b(x_{i_m}))$-rational, we end the trail there, and let $l = m$. Otherwise, from the definition of blocking trails, for $f_{m+1} = b(x_{i_m})$, the ending subsequence $T_f^{\geq m} = \{x_m, ..., x_M\} \cap T_f$ is (A, f_{m+1})-rational. Using Lemma A.1, there is a contract $x_{i_{m+1}} \in T_f^{\geq m} \cap X_f^S$ such that $i_{m+1} > i_m$ and $\{x_{i_{m-1}}, x_{i_m}\}$ is a (A, f_m)-rational pair.

This way, we constructed a locally blocking trail, therefore A is not fully trail-stable.

To show that every set-stable outcome is fully trail-stable, consider an outcome A which is not fully trail-stable, and choose the shortest locally blocking trail T for it. For every firm involved in T, if $T_f \nsubseteq C^f(A \cup T_f)$, then using Lemma A.2 there is a upstream-downstream contract-pair $x_j \in T$ and $z_l \in T$ such that $j \neq l$ and $\{x_j, z_l\}$ is (A, f)-rational. This way we get a shorter locally blocking trail or circuit. Since this was the shortest trail, it must be a circuit. Repeat finding shortcuts until we get a circuit Z such that $Z_f \subseteq C^f(A \cup Z_f)$ for every firm f, so this a blocking set. Since $T \cap A = \emptyset$ and $Z \subseteq T$, $Z \cap A = \emptyset$. □

Proof of Proposition 5. Lemma 5 implies that if outcome A is fully trail-stable then A is also trail-stable. So assume that outcome A is trail-stable. If A is not fully trail-stable then there is a locally blocking trail T to A. The separability property of the choice functions imply that T is a blocking trail, contradicting the trail stability of A. So A is fully trail-stable. □

Proof of Proposition 6. Lemma 5 implies that if outcome A is set-stable then A is also fully trail-stable. Assume that outcome A is fully trail-stable, but not set-stable, so it has a blocking set Z.

Case 1: For every $z \in Z$, contract z is neither $(A, s(z))$-rational nor $(A, b(z))$-rational. Then using Lemma A.1 we can find a circuit $Q = \{z_1, z_2, \dots z_k\} \subseteq Z$ such that $\{z_i, z_{i+1}\}$ is an $(A, b(z_i))$-rational pair for every $1 \leq i \leq k$ and $\{z_k, z_1\}$ is an $(A, b(z_k))$-rational pair. Since every $\{z_i, z_{i+1}\}$ an $(A, b(z_i))$-rational set by itself, as choice functions are simple, intensity function w must strictly decrease along circuit Q, which is impossible.

Case 2: If some contracts $z \in Z$ are A-rational. Suppose that z_1 is $(A, s(z_1))$-rational. From Lemma A.1 we can find a trail $\{z_2, z_3 \dots z_k\} \subseteq Z$ such that for every z_i, either $\{z_i, z_{i+1}\}$ is a $(A, b(z_i))$-rational pair, (therefore $w(z_i) > w(z_{i+1})$) or there are some $y_1 \dots y_l$ such that $b(y_j) = s(z_i)$ for all $1 \leq j \leq l$ and $\{z_i, y_1 \dots y_l\}$ is $(A, b(z_i))$-rational. From the simplicity property there is a y_j such that $w(z_i) > w(y_j)$, this y_j contract will be z_{i+1}. The trail terminates at the first occasion when z_i is $(A, b(z_i))$-rational.

Since the intensity strictly decreases, we cannot get back to a contract used earlier in the trail, so the trail must terminate. Let us pick a contract z_i in the trail such that it is the last one which is $(A, s(z_i))$-rational, and then choose the smallest j such that $j \geq i$ and z_j is $(A, b(z_j))$-rational. From Lemma A.1, the trail from z_i to z_j is locally blocking, so outcome A is not fully trail-stable. $\square$

A.4 Proofs of Lemma 3 and Lemma 4

A.4.1 The sublattice property of fixed points

We can rephrase the definitions of the Laws of Aggregate Demand and Supply (LAD/LAS) in the following way:

If the choice functions of firm f satisfy LAD/LAS, for sets of contracts $Y' \subseteq Y \subseteq X_f^B$, and $Z \subseteq Z' \subseteq X_f^S$ (i.e. $(Y', Z') \sqsubseteq (Y, Z)$) then $|C_B^f(Y'|Z')| - |C_S^f(Z'|Y')| \leq |C_B^f(Y|Z)| - |C_S^f(Z|Y)|$.

For every firm f we define a weight function on the contracts in X_f, namely let $w(x) = 1$ if $x \in X_f^B$ and $w(x) = -1$ if $x \in X_f^S$. So $w(C^f(Y, Z)) = |C_B^f(Y|Z)| - |C_S^f(Z|Y)|$. Therefore if C^f satisfies LAD/LAS, then $(Y', Z') \sqsubseteq (Y, Z)$ implies $w(C^f(Y', Z')) \leq w(C^f(Y, Z))$.

Let Y and Y' be subsets of X_f^B, Z and Z' are subsets of X_f^S. We denote the complement of Z in X_f^S with $\overline{Z} = X_f^S \setminus Z$. Define the operation $(Y, Z)\widetilde{\setminus}(Y', Z') = (Y \setminus Y', \overline{Z' \setminus Z})$. For a given firm f, we call a set function $R : 2^X_f \to 2^X_f$ a *w-contraction* if for every $(Y', Z') \sqsubseteq (Y, Z)$ pair, $w(R(Y, Z)\widetilde{\setminus}R(Y', Z')) \leq w((Y, Z)\widetilde{\setminus}(Y', Z'))$

Let us describe some properties of this $\widetilde{\setminus}$ operation:

Lemma A.3. For a firm f, let Y and Y' be subsets of X_f^B, Z and Z' are subsets of X_f^S such that $(Y', Z') \sqsubseteq (Y, Z)$. Then the following holds:

1. $w((Y, Z)\widetilde{\setminus}(Y', Z')) = w((Y, Z)) - w((Y', Z')) - |X_f^S|$.
2. For any (A, B) pair, $w((A, B)\widetilde{\setminus}(Y, Z)) \leq w((A, B)\widetilde{\setminus}(Y', Z'))$.
3. If $(Y, Z) \sqsubseteq (A, B)$ then the $w((A, B)\widetilde{\setminus}(Y, Z)) = w((A, B)\widetilde{\setminus}(Y', Z'))$ equality implies $(Y', Z') = (Y, Z)$.

Proof. Let us tackle each statement separately:

1. $w((Y, Z)\widetilde{\setminus}(Y', Z')) = |Y \setminus Y'| - |\overline{Z' \setminus Z}| = |Y| - |Y'| - |X_f^S| + |Z'| - |Z| = w((Y, Z)) - w((Y', Z')) - |X_f^S|$.

2. Since $Y \supseteq Y'$, this implies $A \setminus Y \subseteq A \setminus Y'$, and similarly $Z \subseteq Z'$ gives us $Z \setminus B \subseteq Z' \setminus B$, so $\overline{Z \setminus B} \supseteq \overline{Z' \setminus B}$, therefore $w((A,B)\widetilde{\setminus}(Y,Z)) = |A \setminus Y| - |\overline{Z \setminus B}| \leq |A \setminus Y'| - |\overline{Z' \setminus B}| = w((A,B)\widetilde{\setminus}(Y',Z'))$

3. If $w((A,B)\widetilde{\setminus}(Y,Z)) = w((A,B)\widetilde{\setminus}(Y',Z'))$ then equality must hold at $|A \setminus Y| = |A \setminus Y'|$ and $|\overline{Z \setminus B}| = |\overline{Z' \setminus B}|$. Since $Y' \subseteq Y \subseteq A$ and $Z' \supseteq Z \supseteq B$, we get that $Y = Y'$ and $Z = Z'$.

□

Lemma A.4. For a given firm f, if the firm's choice functions are fully substitutable and satisfy LAD/LAS, then the rejection function R^f is $\sqsubseteq$-isotone and a w-contraction.

Proof. Let Y and Y' be subsets of X^B_f and Z and Z' are be of X^S_f, and moreover let $(Y', Z') \sqsubseteq (Y, Z)$.
We have seen earlier that R^f is $\sqsubseteq$-isotone, so $R^f(Y',Z') \sqsubseteq R^f(Y,Z)$. To prove that it is w-contraction, $w(R^f(Y,Z)\widetilde{\setminus}R^f(Y',Z')) + |X^S_f| = w(R^f(Y,Z)) - w(R^f(Y',Z')) = w((Y,Z) \setminus C^f(Y,Z)) - w((Y',Z') \setminus C^f(Y',Z')) = w(Y,Z) - w(C^f(Y,Z)) - w(Y',Z') + w(C^f(Y',Z')) \leq w(Y,Z) - w(Y',Z') = w((Y,Z)\widetilde{\setminus}(Y',Z')) + |X^S_f|$. We used that $w(C^f(Y',Z')) \leq w(C^f(Y,Z))$. If we subtract $|X^S_f|$ from both sides, we get that
$w(R^f(Y,Z)\widetilde{\setminus}R^f(Y',Z')) \leq w((Y,Z)\widetilde{\setminus}(Y',Z'))$, so R^f is indeed a w-contraction. □

We will work on the $(2^{(X,X)}, \widetilde{\cup}, \widetilde{\cap})$ lattice. We can imagine it as a network that contains exactly two (unrelated) copies of each contract, one for the buyer and one for the seller of the contract.

Now the C^f choice functions of the firms are defined over disjoint set of contracts, so for every $Y \subseteq (X,X)$ we can define $C(Y) = \bigcup_{f \in F} C^f(Y_f)$. Similarly $R(Y) = \bigcup_{f \in F} R^f(Y_f)$. On this whole network, we call a set function $R : 2^{(X,X)} \to 2^{(X,X)}$ a *w-contraction* if for every firm f the corresponding R_f was a w-contraction.

Let us denote the set of the starting half-contracts (seller's side) with $X^S_F = \bigcup_{f \in F} X^S_f$, and the set of ending half-contracts (buyer's side) with $X^B_F = \bigcup_{f \in F} X^B_f$. Now $|X^S_F| = |X^B_F| = |X|$.

Let $Y \subseteq X^B_F$ and $Z \subseteq X^S_F$. The *dual* of (Y,Z) is what we get by switching the two parts. We denote it with $(Y,Z)^* = (Z,Y)$.

In this model let all the contracts in X^S_F have weight $w = -1$ and all contracts in X^B_F have weight $w = 1$.

Lemma A.5. If $F : 2^{(X,X)} \to 2^{(X,X)}$ is $\sqsubseteq$-isotone and a w-contraction then fixed points of F form a nonempty sublattice of $(2^{(X,X)}, \widetilde{\cup}, \widetilde{\cap})$

Proof. By Tarski's fixed-point theorem, the set of fixed points is nonempty. Now let $U \subseteq (X,X)$ and $V \subseteq (X,X)$. Assume that $F(U) = U$ and $F(V) = V$. By monotonicity, $U \widetilde{\cap} V = F(U) \widetilde{\cap} F(V) \sqsupseteq F(U \widetilde{\cap} V)$ and $U \widetilde{\cup} V = F(U) \widetilde{\cup} F(V) \sqsubseteq F(U \widetilde{\cap} V)$. From the w-contraction property and Lemma A.3

$$w(U\widetilde{\setminus}(U \widetilde{\cap} V)) \geq w(F(U)\widetilde{\setminus}F(U \widetilde{\cap} V)) \geq w(U\widetilde{\setminus}(U \widetilde{\cap} V))$$

$$w((U \mathbin{\widetilde{\cup}} V) \mathbin{\widetilde{\setminus}} U) \geq w(F(U \mathbin{\widetilde{\cup}} V) \mathbin{\widetilde{\setminus}} F(U)) \geq w((U \mathbin{\widetilde{\cup}} V) \mathbin{\widetilde{\setminus}} U)$$

hence there must be equality throughout. Using the third part of Lemma A.3 we can see that $(U \mathbin{\widetilde{\cap}} V) = F(U \mathbin{\widetilde{\cap}} V)$ and $(U \mathbin{\widetilde{\cup}} V) = F(U \mathbin{\widetilde{\cup}} V)$ so they are also fixed points of F. ☐

Observation A.2. Consider two sets of contracts (Y, Z) and (Y', Z') , where $Y, Y' \subseteq X_F^B$ and $Z, Z' \subseteq X_F^S$ and $(X, X) \setminus (Y, Z) = (X \setminus Y, X \setminus Z)$. If $(Y', Z') \sqsubseteq (Y, Z)$, then $((X \setminus Y, X \setminus Z)^* \mathbin{\widetilde{\setminus}} (X \setminus Y', X \setminus Z')^*) = ((X \setminus Z) \setminus (X \setminus Z'), \overline{(X \setminus Y') \setminus (X \setminus Y)}) = ((Z' \setminus Z), \overline{(Y \setminus Y')}) = ((X, X) \setminus ((Y, Z) \mathbin{\widetilde{\setminus}} (Y', Z'))^*$

Theorem A.3. *If the choice functions of all agents are fully substitutable and satisfy LAD/LAS, then the fixed points of $\Phi(Y, Z) = (X \setminus R_S(Z|Y), X \setminus R_B(Y|Z))$ form a nonempty, complete sublattice of $(2^X \times 2^X, \widetilde{\cup}, \widetilde{\cap})$.*

Proof. The $\Phi(Y, Z) = (X \setminus R_S(Z|Y), X \setminus R_B(Y|Z))$ function can be also written as $\Phi(Y) = ((X, X) \setminus R(Y, Z))^*$. Since R is $\sqsubseteq$-isotone, Φ is also $\sqsubseteq$-isotone. We need to show that Φ is a w-contraction. Suppose that $(Y', Z') \sqsubseteq (Y, Z)$. Using Observation A.2, $w(\Phi(Y, Z) \mathbin{\widetilde{\setminus}} \Phi(Y', Z')) = w(((X, X) \setminus R(Y, Z))^* \mathbin{\widetilde{\setminus}} ((X, X) \setminus R(Y', Z'))^*) = w(((X, X) \setminus (R(Y, Z) \mathbin{\widetilde{\setminus}} R(Y', Z')))^*) = w(R(Y, Z) \mathbin{\widetilde{\setminus}} R(Y', Z'))$ $w((Y, Z) \mathbin{\widetilde{\setminus}} (Y', Z'))$ because in Lemma A.4 we showed that R is a w-contraction.

Since Φ is $\sqsubseteq$-isotone and a w-contraction, Lemma A.5 gives that the fixed points of Φ form a sublattice of $(2^{(X,X)}, \widetilde{\cup}, \widetilde{\cap})$. ☐

A.4.2 Lattice for the terminal agents

Lemma A.6. (Path Independence) If choice function $C : 2^X \to 2^X$ is same-side substitutable and satisfies IRC then $C(Y \cup Z) = C(Y \cup C(Z))$ holds for any subsets Y, Z of X.

Proof. Since C is same-side substitutable, $C(Y \cup Z) \subseteq (Y \cup C(Z))$. Using IRC we have that $C(Y \cup Z) \subseteq (Y \cup C(Z)) \subseteq (Y \cup Z)$ implies that $C(Y \cup Z) = C(Y \cup C(Z))$. ☐

Lemma A.7. For a set of contracts X, if choice functions of F are fully substitutable and satisfy IRC, then terminal superiority is a partial order on terminal-fully-trail-stable outcomes.

Proof of Lemma A.7. We need to prove that $\preceq^S$ is reflexive, antisymmetric and transitive. Assume that A, A' and A'' are acceptable outcomes. As $C^f(A_f \cup A_f) = C^f(A_f) = A_f$ holds for each agent (and hence for each terminal seller) f, relation $\preceq^S$ is reflexive. If $A \preceq^S A' \preceq^S A$ then we have $A_f = C^f(A_f \cup A'_f) = A'_f$ holds for any terminal agent f, hence $A = A'$ and $\preceq^S$ is antisymmetric. For transitivity, assume that $A \succeq^S A' \succeq^S A''$. Using this and Lemma A.6, we get for any terminal agent f that

$$C^f(A_f \cup A''_f) = C^f(C^f(A_f \cup A'_f) \cup A''_f) = C^f(A_f \cup A'_f \cup A''_f) = C^f(A_f \cup C^f(A'_f \cup A''_f)) = C^f(A_f \cup A'_f) = A_f\ ,$$

hence $A \succeq^S A''$ holds, indeed. This completes the proof. ☐

Theorem A.4. *If L is a nonempty complete sublattice of $(2^X \times 2^X, \widetilde{\cup}, \widetilde{\cap})$ then $L_{\mathcal{T}} = \{(Y_{\mathcal{T}}, Z_{\mathcal{T}}) : (Y, Z) \in L\}$ is a sublattice of $(2^{\mathcal{T}} \times 2^{\mathcal{T}}, \widetilde{\cup}, \widetilde{\cap})$.*

Proof. For a given $(A_{\mathcal{T}}, B_{\mathcal{T}})$ there can be more than one inverse image in the original lattice. since L is a complete lattice with lattice operations $\widetilde{\cup}, \widetilde{\cap}$, this $(A^*, B^*) \in L$ and $(A^*_{\mathcal{T}}, B^*_{\mathcal{T}}) = (A_{\mathcal{T}}, B_{\mathcal{T}})$. We call it the *canonical inverse image* of $(A_{\mathcal{T}}, B_{\mathcal{T}})$, since this is the $\sqsubseteq$-greatest among all inverse images.

If $(A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(C_{\mathcal{T}}, D_{\mathcal{T}}) \in L_{\mathcal{T}}$, let us consider $(Y, Z) = (A^*, B^*) \widetilde{\cap} (C^*, D^*)$. Since $(Y, Z) \sqsubseteq (A^*, B^*)$ this implies $(Y_{\mathcal{T}}, Z_{\mathcal{T}}) \sqsubseteq (A^*_{\mathcal{T}}, B^*_{\mathcal{T}}) = (A_{\mathcal{T}}, B_{\mathcal{T}})$. Similarly $(Y_{\mathcal{T}}, Z_{\mathcal{T}}) \sqsubseteq (C_{\mathcal{T}}, D_{\mathcal{T}})$. We want to show that $(Y_{\mathcal{T}}, Z_{\mathcal{T}})$ is the greatest lower bound of $(A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(C_{\mathcal{T}}, D_{\mathcal{T}})$ in $L_{\mathcal{T}}$. We can see that $(Y^*, Z^*) \sqsubseteq (A^*, B^*)$ and $(Y^*, Z^*) \sqsubseteq (C^*, D^*)$ because (A^*, B^*) is defined by the union of a greater set. Therefore $(Y^*, Z^*) = (Y, Z)$.

Suppose there exists a $(E_{\mathcal{T}}, F_{\mathcal{T}}) \in L_{\mathcal{T}}$ such that $(E_{\mathcal{T}}, F_{\mathcal{T}}) \sqsubseteq (A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(E_{\mathcal{T}}, F_{\mathcal{T}}) \sqsubseteq (C_{\mathcal{T}}, D_{\mathcal{T}})$ but $(E_{\mathcal{T}}, F_{\mathcal{T}}) \not\sqsubseteq (Y_{\mathcal{T}}, Z_{\mathcal{T}})$. Then in the original lattice $(E^*, F^*) \sqsubseteq (A^*, B^*)$ and $(E^*, F^*) \sqsubseteq (C^*, D^*)$ but $(E^*, F^*) \not\sqsubseteq (Y^*, Z^*)$. But this is impossible since $(Y, Z) = (A^*, B^*) \widetilde{\cap} (C^*, D^*)$. So we have found a unique greatest common lower bound of $(A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(C_{\mathcal{T}}, D_{\mathcal{T}})$.

Similar argument can be made in order to find the lowest common upper bound of $(A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(C_{\mathcal{T}}, D_{\mathcal{T}})$. Let $(Y, Z) = (A^*, B^*) \widetilde{\cup} (C^*, D^*)$. Since $(Y, Z) \sqsupseteq (A^*, B^*)$ this implies $(Y_{\mathcal{T}}, Z_{\mathcal{T}}) \sqsupseteq (A^*_{\mathcal{T}}, B^*_{\mathcal{T}}) = (A_{\mathcal{T}}, B_{\mathcal{T}})$. Similarly $(Y_{\mathcal{T}}, Z_{\mathcal{T}}) \sqsupseteq (C_{\mathcal{T}}, D_{\mathcal{T}})$.

Suppose there exists a $(E_{\mathcal{T}}, F_{\mathcal{T}}) \in L_{\mathcal{T}}$ such that $(E_{\mathcal{T}}, F_{\mathcal{T}}) \sqsupseteq (A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(E_{\mathcal{T}}, F_{\mathcal{T}}) \sqsupseteq (C_{\mathcal{T}}, D_{\mathcal{T}})$ but $(E_{\mathcal{T}}, F_{\mathcal{T}}) \not\sqsupseteq (Y_{\mathcal{T}}, Z_{\mathcal{T}})$. Then in the original lattice $(E^*, F^*) \sqsupseteq (A^*, B^*)$ and $(E^*, F^*) \sqsupseteq (C^*, D^*)$ therefore $(E^*, F^*) \sqsupseteq (Y, Z)$, so $(E^*_{\mathcal{T}}, F^*_{\mathcal{T}}) = (E_{\mathcal{T}}, F_{\mathcal{T}}) \sqsupseteq (Y_{\mathcal{T}}, Z_{\mathcal{T}})$, which is a contradiction.

So we have found a unique lowest common upper bound of $(A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(C_{\mathcal{T}}, D_{\mathcal{T}})$, so $(L_{\mathcal{T}}, \widetilde{\cup}, \widetilde{\cap})$ is indeed a lattice. $\square$

Now we consider only the contracts sold by the terminal sellers. For any $Y \subseteq X$, let $Y_{\mathcal{S}} = \{x \in Y | s(x) \in \mathcal{T}\}$.

Given two fully trail-stable outcomes A and A', let us denote the canonical stable pair (defined as at the end of Proof of Lemma 2) for A with $\dot{X}^B$ and $\dot{X}^S$, and the canonical stable pair for A' with $\dot{X}'^B$ and $\dot{X}'^S$.

Lemma A.8. Given two fully trail-stable outcomes A and A', $C^f(A_f \cup A'_f) = A_f$ for each teminal seller if and only if $\dot{X}^S_{\mathcal{S}} \supseteq \dot{X}'^S_{\mathcal{S}}$ and $\dot{X}^B_{\mathcal{S}} \subseteq \dot{X}'^B_{\mathcal{S}}$ holds. A similar statement holds for terminal buyers.

Proof. If f is a terminal seller, $C^f(\dot{X}^S) = A_f$ and $C^f(\dot{X}'^S) = A'_f$. Suppose that $\dot{X}^S_{\mathcal{S}} \supseteq \dot{X}'^S_{\mathcal{S}}$. By IRC, $A_f \subseteq A_f \cup A'_f \subseteq \dot{X}^S_f$ implies that $C^f(A_f \cup A'_f) = A_f$.

For the opposite direction, take a contract $x \in X_f$ such that $x \notin C^f(A'_f \cup x)$. We use Lemma A.7, $A \succeq^S A' \succeq^S x$, therefore $A \succeq^S x$, so $x \notin C^f(A_f \cup x)$. When we define the stable pairs for A and A', if $x \in C^f(A'_f \cup x)$ then $x \in \dot{X}^B$, if $x \notin C^f(A'_f \cup x)$ then $x \in \dot{X}^S$. From the previous observation we can see that $\dot{X}^S_{\mathcal{S}} \supseteq \dot{X}'^S_{\mathcal{S}}$ and $\dot{X}^B_{\mathcal{S}} \subseteq \dot{X}'^B_{\mathcal{S}}$. The proof for terminal buyers is analogous. $\square$

Proof of Lemma 3. In the proof of Lemma 2 we have seen that any fixed point $(\dot{X}^B, \dot{X}^S)$ of isotone mapping Φ on lattice L determines a fully-trail-stable outcome A^X. Moreover, each stable outcome A corresponds to at least one fixed point $(\dot{X}^B, \dot{X}^S)$ of Φ. From Theorem A.1, it follows that fixed points of Φ form a lattice, hence there is a $\sqsubseteq$-minimal fixed point $(\dot{Y}^B, \dot{Y}^S)$ and a $\sqsubseteq$-maximal one $(\dot{Z}^B, \dot{Z}^S)$. We show that fully-trail-stable outcome A^Y is seller-optimal and A^Z is buyer-optimal. So assume that $A = A^X$ is a fully-trail-stable outcome. As $(\dot{Y}^B, \dot{Y}^S) \sqsubseteq (\dot{X}^B, \dot{X}^S) \sqsubseteq (\dot{Z}^B, \dot{Z}^S)$, we have $\dot{Y}^B \subseteq \dot{X}^B \subseteq \dot{Z}^B$ and $\dot{Y}^S \supseteq \dot{X}^S \supseteq \dot{Z}^S$. Lemma A.8 implies that $C^f(A_f \cup A_f^Y) = A_f^Y$ and $C^f(A_f \cup A_f^Z) = A_f$ for any terminal seller f and $C_g(A_g \cup A_g^Y) = A_g$ and $C_g(A_g \cup A_g^Z) = A_g^Z$ for any terminal buyer g. So, by definition A is seller-superior to A^Y and A^Z is seller-superior to A. □

Proof of Lemma 4. In the proof of Lemma 2 we have seen that A is fully trail-stable if and only if there is canonical pair $(\dot{X}^B, \dot{X}^S)$ of such that $(\dot{X}^B, \dot{X}^S)$ is a fixed point of isotone mapping Φ and $A = \dot{X}^B \cap \dot{X}^S$. Moreover, if the choice functions satisfy LAD/LAS, then fixed points of Φ form a sublattice L of $(2^X \times 2^X, \widetilde{\cup}, \widetilde{\cap})$. From Lemma A.4, the projection of the above lattice to the terminals, $L_{\mathcal{T}}$ is also a lattice under $\sqsubseteq$ and from Lemma A.8 this partial order coincides with $\preceq^S$. Therefore, the fully trail-stable outcomes form a lattice under terminal-superiority. □

A.5 Proofs of Propositions 3 and 4

Our proof is similar to Ostrovsky's proof. First we investigate the restabilized outcome from A, which we play part in the proofs of both Propositions 3 and 4. Let A be an arbitrary fully trail-stable outcome in the original network, with a correspondong canonical pair $(\dot{X}^B, \dot{X}^S)$. After the new terminal seller f' arrives, let X be the set of all contracts in the new network, and let us define $(X^{*B}, X^{*S}) = (\dot{X}^B, \dot{X}^S \cup X_{f'})$. In the following, we will use Φ according to the choice fuctions on the new network, so $(\dot{X}^B, \dot{X}^S)$ does not need to be a fixed point of Φ anymore.

Since $X_{f'} \cap X^{*B} = \emptyset$, for every firm $f \neq f'$, $R_S^f(X^{*S}|X^{*B}) = R_S^f(\dot{X}^S|\dot{X}^B)$ and $R_B^f(X^{*B}|X^{*S}) = R_B^f(\dot{X}^B|\dot{X}^S)$. For example, if f has a conctracts with f', contract $x = f'f$ was not offered for firm f in X^{*B} so it does not get rejected.

For firm f', $R_S^{f'}(X^{*S}|X^{*B}) = X_{f'} \setminus C_S^{f'}(X_{f'})$ and $R_B^{f'}(X^{*B}|X^{*S}) = \emptyset$.

Therefore $\Phi(X^{*B}, X^{*S}) = (\dot{X}^B \cup C^{f'}(X_{f'}), \dot{X}^S \cup X_{f'})$.

So $(X^{*B}, X^{*S}) \sqsubseteq \Phi(X^{*B}, X^{*S})$, and Φ is $\sqsubseteq$-isotone, so $\Phi(X^{*B}, X^{*S}) \sqsubseteq \Phi(\Phi(X^{*B}, X^{*S}))$ and so on. The lattice of all possible subset-pairs is finite, so there is a k such that $\Phi^k(X^{*B}, X^{*S}) = (\hat{X}^B, \hat{X}^S)$ is a fixed point. So $(X^{*B}, X^{*S}) \sqsubseteq \Phi(X^{*B}, X^{*S}) \sqsubseteq \Phi^k(X^{*B}, X^{*S}) = (\hat{X}^B, \hat{X}^S)$. Outcome $\hat{A} = \hat{X}^B \cap \hat{X}^S$ is fully trail-stable, and this is what we call the *restabilized outcome* from A.

Proof of Proposition 3. If f' is a terminal seller, and we start from outcome A_{max} and the $\sqsubseteq$-maximal pair $(\dot{Z}^B, \dot{Z}^S)$. Using the previous method, outcome $\hat{A} = \hat{Z}^B \cap \hat{Z}^S$ is the restabilized outcome from A. In the new network there exists a $\sqsubseteq$-maximal fixed point of Φ, namely (Z'^B, Z'^S), therefore $(\dot{Z}^B, \dot{Z}^S \cup X_{f'}) = (Z^{*B}, Z^{*S}) \sqsubseteq (\hat{Z}^B, \hat{Z}^S) \sqsubseteq (Z'^B, Z'^S)$. The fully trail-stable outcome corresponding to the maximal fixed point is $A'_{max} = Z'^B \cap Z'^S$. We have to show that A'_{max} is better for terminal buyers and worse for terminal sellers than the original A_{max}. If f is a terminal buyer, since (Z'^B, Z'^S) is fixed point of Φ and $(\dot{Z}^B, \dot{Z}^S)$ was fixed before the new agent arrived, $C^f(Z'^B) = A'_{f,max}$ and $C^f(Z^{*B}) = A_{f,max}$ and $Z^{*B} \subseteq Z'^B$ so from $C^f(Z'^B) \subseteq (A_{f,max} \cup A'_{f,max}) \subseteq Z'^B$ by IRC we obtain $C^f(A_{f,max} \cup A'_{f,max}) = A'_{f,max}$ so $A'_{f,max}$ is preferred by terminal buyers.

Similarly, if f is a terminal seller outside f', $C^f(Z'^S) = A'_{f,max}$ and $C^f(Z^{*S}) = A_{f,max}$ and $Z'^S \subseteq Z^{*S}$ so from $C^f(Z^{*S}) \subseteq (A_{f,max} \cup A'_{f,max}) \subseteq Z^{*S}$ by IRC we obtain $C^f(A_{f,max} \cup A'_{f,max}) = A_{f,max}$ so $A_{f,max}$ is preferred by terminal buyers. If f' is a terminal buyer then we can use the same proof with reversing the roles of buyers and sellers. $\square$

Proof of Proposition 4. If f' is a terminal seller, and A is any fully trail-stable outcome in the original network, with canonical pair $(\dot{X}^B, \dot{X}^S)$, then $(X^{*B}, X^{*S}) = (\dot{X}^B, \dot{X}^S \cup X_{f'}) \sqsubseteq (\hat{X}^B, \hat{X}^S)$. The restabilized outcome is $\hat{A} = \hat{X}^B \cap \hat{X}^S$, and similarly to the proof of Proposition 3 one can show that initial producers weakly prefer A to $\hat{A}$ and all end consumers (other than f') prefer $\hat{A}$ to A. If f' is a terminal buyer, preferences are the opposite. $\square$

A.6 Proof of Theorem 3

Proof of Theorem 3. Consider the price adjustment process in a two-sided many-to-many market with contracts and prices introduced by Roth (1984). We take the perspective of buyers, i.e. buying firms are offering their upstream trades to corresponding sellers. We initialize the price adjustment process at the lowest possible prices so in the first step every upstream contract is offered (CP1). When the price adjustment process terminates (by Tarski fixed-point theorem, it must), we are at a (fully) trail-stable outcome A (by Theorem 2). Note that in this price adjustment process:

- Prices are specified for each trade at every step.
- Prices of all trades are not decreasing in every round.
- Offers remain open. If a seller accepts a trade at round t, the buyer will offer it at all subsequent rounds.
- Rejections remain final. If a seller rejects a trade at round t, he will reject it at all subsequent rounds.

We tracked a price for every trade. For trades $\Psi = \tau(A)$ that are realized at the trail-stable outcome A, we assign prices specified in A to those trades. Clearly, these trades are chosen at these prices since the corresponding contracts are chosen. If $\tau(A) = \Omega$, this indeed a competitive equilibrium.

If a trade ω has not been realized, then it must have not been in $b(\omega)$'s chosen set in the final round T (otherwise both buyer and seller would choose the trade at price $\check{p}_\omega$). That means it was rejected by $s(\omega)$ in some round $t < T$ at a lower price p^*_ω and this price has not changed (by (CP1), (PM)). Since prices of other trades have increased and since rejections remain final, $s(\omega)$ will continue rejecting this trade at p^*_ω at T. Using (CP3), we can find a price $\tilde{p}_\omega$ for every unrealized trade one by one such that the trade is rejected by $b(\omega)$ and $s(\omega)$. Assign some price $\tilde{p}_\omega$ to all such trades. Note that this does not affect the choice of other contracts (since prices are adjusted weakly downward for buyer and weakly upward for the seller and they continue to reject the particular trade; adding rejected trades is irrelevant to choices). Now all trades have been assigned prices giving us a set of contracts $\kappa([\Omega^*, p^*])$ where $p^*_\omega = p_\omega$ for $(\omega, p_\omega) \in A$ and $\tilde{p}$ otherwise. At these prices agents only choose contracts they were allocated at A; the realized trades Ψ. Hence, this is a competitive equilibrium and (full) trail stability is preserved. This completes the proof. $\square$

References

Acemoglu, D., V. M. Carvalho, A. Ozdaglar, and A. Tahbaz-Salehi (2012). The network origins of aggregate fluctuations. *Econometrica 80*(5), 1977–2016.

Adachi, H. (2000). On a characterization of stable matchings. *Economics Letters 68*(1), 43–49.

Alaei, S., K. Jain, and A. Malekian (2011). Competitive equilibrium in two sided matching markets with general utility functions. *ACM SIGecom Exchanges 10*(2), 34–36.

Alkan, A. (2002). A class of multipartner matching markets with a strong lattice struture. *Economic Theory 19*(4), 737–746.

Alva, S. (2015a). Pairwise stability and complementarity in matching with contracts. Technical report, University of Texas - San Antonio.

Alva, S. (2015b, November). Warp and combinatorial choice in matching. Technical report, University of Texas - San Antonio.

Alva, S. and A. Teytelboym (2015). Matching in supply chains with complementary inputs. Technical report, In preparation.

Antràs, P. and D. Chor (2013). Organizing the global value chain. *Econometrica 81*(6), 2127–2204.

Aygün, O. and T. Sönmez (2013). Matching with contracts: Comment. *American Economic Review 103*(5), 2050–2051.

Baldwin, E. and P. Klemperer (2013). Tropical geometry to analyse demand. Discussion paper, University of Oxford, Department of Economics.

Bando, K. (2012). Many-to-one matching markets with externalities among firms. *Journal of Mathematical Economics 48*(1), 14 – 20.

Blair, C. (1988). The lattice structure of the set of stable matchings with multiple partners. *Mathematics of Operations Research 13*(4), 619–628.

Blum, Y., A. E. Roth, and U. G. Rothblum (1997). Vacancy chains and equilibration in senior-level labor markets. *Journal of Economic Theory 76*(2), 362–411.

Chakraborty, A., A. Citanna, and M. Ostrovsky (2010). Two-sided matching with interdependent values. *Journal of Economic Theory 145*(1), 85–105.

Crawford, V. P. (1991). Comparative statics in matching markets. *Journal of Economic Theory 54*(2), 389–400.

Crawford, V. P. and E. M. Knoer (1981). Job matching with heterogeneous firms and workers. *Econometrica 49*(2), 437–450.

Demange, G. and D. Gale (1985). The strategy structure of two-sided matching markets. *Econometrica 53*(4), 873–888.

Drexl, M. (2013). Substitutes and complements in trading networks. Technical report, Mimeo.

Echenique, F. (2007). Counting combinatorial choice rules. *Games and Economic Behavior 58*(2), 231–245.

Echenique, F. (2012). Contracts vs. salaries in matching. *American Economic Review 102*(1), 594–601.

Echenique, F. and J. Oviedo (2006). A theory of stability in many-to-many matching markets. *Theoretical Economics 1*(1), 233–273.

Ehlers, L. and J. Massó (2007). Incomplete information and singleton cores in matching markets. *Journal of Economic Theory 136*(1), 587–600.

Fazzari, S., R. G. Hubbard, and B. C. Petersen (1988). Financing constraints and corporate investment. Working Paper 2387, NBER.

Fleiner, T. (2003). A fixed-point approach to stable matchings and some applications. *Mathematics of Operations Research 28*(1), 103–126.

Fleiner, T. (2009). On stable matchings and flows. Technical Report TR-2009-11, Egerváry Research Group, Budapest. `www.cs.elte.hu/egres`.

Fleiner, T. (2014). On stable matchings and flows. *Algorithms 7*, 1–14.

Fox, J. T. (2010, October). Identification in matching games. *Quantitative Economics 1*(2), 203–254.

Frank, J. (1990). Monopolistic competition, risk aversion, and equilibrium recessions. *Quarterly Journal of Economics 105*(4), 921–938.

Gale, D. (1984). Equilibrium in a discrete exchange economy with money. *International Journal of Game Theory 13*(1), 61–64.

Gale, D. and L. S. Shapley (1962). College admissions and the stability of marriage. *American Mathematical Monthly 69*(1), 9–15.

Gale, D. and M. Sotomayor (1985). Some remarks on the stable matching problem. *Discrete Applied Mathematics 11*(3), 223–232.

Hatfield, J. W. and F. Kojima (2009). Group incentive compatibility for matching with contracts. *Games and Economic Behavior 67*, 745–749.

Hatfield, J. W. and S. D. Kominers (2012). Matching in networks with bilateral contracts. *American Economic Journal: Microeconomics 4*(1), 176–208.

Hatfield, J. W. and S. D. Kominers (2013). Vacancies in supply chain networks. *Economics Letters 119*(3), 354–357.

Hatfield, J. W. and S. D. Kominers (2015a, June). Contract design and stability in matching markets. Working paper, Harvard Business School.

Hatfield, J. W. and S. D. Kominers (2015b). Multilateral matching. *Journal of Economic Theory 156*, 175–206.

Hatfield, J. W., S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp (2013). Stability and competitive equilibrium in trading networks. *Journal of Political Economy 121*(5), 966–1005.

Hatfield, J. W., S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp (2015, April). Chain stability in trading networks. Working paper, Mimeo.

Hatfield, J. W. and P. Milgrom (2005). Matching with contracts. *American Economic Review 95*(4), 913–935.

Herings, P. J.-J. (2015). Equilibrium and matching under price controls. Research Memorandum RM/15/001, Maastricht University, Graduate School of Business and Economics (GSBE).

Jensen, M. C. (1986). Agency costs of free cash flow, corporate finance, and takeovers. *American Economic Review 76*(2), 323–329.

Kelso, A. S. and V. P. Crawford (1982). Job matching, coalition formation, and gross substitutes. *Econometrica 50*(6), 1483–1504.

Klaus, B. and M. Walzl (2009). Stable many-to-many matchings with contracts. *Journal of Mathematical Economics 45*(7-8), 422–434.

Kominers, S. D. (2012). On the correspondence of contracts to salaries in (many-to-many) matching. *Games and Economic Behavior 75*(2), 984 – 989.

Milgrom, P. and J. Roberts (1990). The economics of modern manufacturing: Technology, strategy, and organization. *American Economic Review 80*(3), 511–528.

Morimoto, S. and S. Serizawa (2015). Strategy-proofness and efficiency with non-quasilinear preferences: a characterization of minimum price Walrasian rule. *Theoretical Economics 10*(2), 445–487.

Ostrovsky, M. (2008). Stability in supply chain networks. *American Economic Review 98*(3), 897–923.

Papadimitriou, C. H. (2007). The Complexity of Finding Nash Equilibria. In N. Nisan, T. Roughgarden, Éva Tardos, and V. V. Vazirani (Eds.), *Algorithmic Game Theory*. Cambridge University Press.

Pycia, M. (2012). Stability and preference alignment in matching and coalition formation. *Econometrica 80*(1), 323–362.

Pycia, M. and M. B. Yenmez (2015, January). Matching with externalities. Technical report, UCLA.

Quinzii, M. (1984). Core and competitive equilibria with indivisibilities. *International Journal of Game Theory 13*(1), 41–60.

Roth, A. (1986). On the allocation of residents to rural hospitals: A general property of two-sided matching markets. *Econometrica 54*, 425–427.

Roth, A. E. (1984). Stability and polarization of interests in job matching. *Econometrica 52*(1), 47–58.

Roth, A. E. (1985). Conflict and coincidence of interest in job matching: some new results and open questions. *Mathematics of Operations Research 10*(3), 379–389.

Roth, A. E. (1989). Two-sided matching with incomplete information about others' preferences. *Games and Economic Behavior 1*(2), 191–209.

Roth, A. E. (1991). A natural experiment in the organization of entry-level labor markets: Regional markets for new physicians and surgeons in the United Kingdom. *American Economic Review 81*(3), 415–440.

Roth, A. E. and M. Sotomayor (1990). *Two-sided Matching: A Study in Game-Theoretic Modeling and Analysis.* Cambridge University Press.

Sasaki, H. and M. Toda (1996). Two-sided matching problems with externalities. *Journal of Economic Theory 70*(1), 93–108.

Sotomayor, M. (1999). Three remarks on the many-to-many stable matching problem. *Mathematical Social Sciences 38*, 55–70.

Sotomayor, M. (2004). Implementation in the many-to-many matching market. *Games and Economic Behavior 46*, 199–212.

Tarski, A. (1955). A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics 5*(2), 285–309.

Teytelboym, A. (2014). Gross substitutes and complements: a simple generalization. *Economics Letters 123*(2), 135–138.

Westkamp, A. (2010). Market structure and matching with contracts. *Journal of Economic Theory 145*(5), 1724–1738.