Trading Networks with Bilateral Contracts*

Tamás Fleiner† Zsuzsanna Jankó‡ Akihisa Tamura§
Alexander Teytelboym¶

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Abstract

We consider a model of matching in trading networks in which firms can enter into bilateral contracts. In trading networks, *stable* outcomes, which are immune to deviations of arbitrary sets of firms, may not exist. We define a new solution concept called *trail stability*. Trail-stable outcomes are immune to consecutive, pairwise deviations between linked firms. We show that any trading network with bilateral contracts has a trail-stable outcome whenever firms' choice functions satisfy the full substitutability condition. For trail-stable outcomes, we prove results on the lattice structure, the rural hospitals theorem, strategy-proofness, and comparative statics of firm entry and exit. We also introduce *weak* trail stability which is implied by trail stability under full substitutability. We describe relationships between the solution concepts.

Keywords: matching markets, market design, trading networks, supply chains, trail stability, weak trail stability, chain stability, stability, contracts.

JEL Classification: *C78, D47, L14*

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†Budapest University of Technology and Economics, Budapest, Hungary. Research was supported by the OTKA K108383 research project and the MTA-ELTE Egerváry Research Group. Part of the research was carried out during two working visits at Keio University. E-mail: `fleiner@cs.bme.hu`

‡Corvinus University, Budapest, Hungary. Research was supported by the OTKA K109240 research project and the MTA-ELTE Egerváry Research Group. E-mail: `jzsuzsy@cs.elte.hu`

§Keio University, Yokohama, Japan. Research was supported by Grants-in-Aid for Scientific Research (B) from JSPS. E-mail: `aki-tamura@math.keio.ac.jp`

¶Department of Economics, St. Catherine's College, and the Institute of New Economic Thinking at the Oxford Martin School, University of Oxford, Oxford, United Kingdom. Email: `alexander.teytelboym@economics.ox.ac.uk`

1 Introduction

Modern production is highly interconnected. Firms typically have a large number of buyers and suppliers and dozens of intermediaries add value to final products before they reach the consumer. In this paper, we study the structure of contractual relationships between firms. In our model, firms have heterogeneous preferences over sets of bilateral contracts with other firms. Contracts may encode many dimensions of a relationship including the price, quantity, quality, and delivery time. The universe of possible relationships between firms is described by a *trading network*—a multi-sided matching market in which firms form downstream contracts to sell outputs and upstream contracts to buy inputs.

We focus on the existence and structure of stable outcomes in decentralized, real-world matching markets. In production networks that we consider in this paper, stable outcomes play the role of equilibrium and may serve as a reasonable prediction of the outcome of market interactions (Kelso and Crawford, 1982, Roth, 1984, Hatfield et al., 2013).[1] We obtain a general result: any trading network has an outcome that satisfies a natural extension of *pairwise stability* (Gale and Shapley, 1962). Our model of matching markets subsumes many previous models of matching with contracts, including many-to-one (Gale and Shapley, 1962, Crawford and Knoer, 1981, Kelso and Crawford, 1982, Hatfield and Milgrom, 2005) and many-to-many matching markets (Roth, 1984, Sotomayor, 1999, 2004, Echenique and Oviedo, 2006, Klaus and Walzl, 2009).

We build on a seminal contribution by Ostrovsky (2008), who introduced a matching model of *supply chains*. In a supply chain, there are agents, who only supply inputs (e.g. farmers); agents, who only buy final outputs (e.g. consumers); while the rest of the agents are intermediaries, who buy inputs and sell outputs (e.g. supermarkets). All agents are partially ordered along the supply chain: downstream (upstream) firms cannot sell to (buy from) firms upstream (downstream) i.e. the trading network is *acyclic*. His key assumption about the market, which we retain in his paper, was that firms' choice functions over contracts satisfy *same-side substitutability* and *cross-side complementarity* conditions (Hatfield and Kominers (2012) later referred to these conditions jointly as *full substitutability*). This assumption requires that firms view any downstream or any upstream contracts as substitutes, but any downstream and any upstream contract as complements.[2] Ostrovsky (2008)

[1] The "market design" literature has emphasized the importance of the existence of stable outcomes in order to prevent centralized matching markets from unraveling (Roth, 1991). One market design application of our model is electricity trading in peer-to-peer power systems (Morstyn et al., 2018).

[2] Same-side substitutability is a fairly strong assumption as, for example, it rules out any complementarities in inputs. There is evidence that modern manufacturing firms rely on many complementary inputs (Milgrom and Roberts, 1990, Fox, 2010). Hatfield and Kominers (2015) and Rostek and Yoder (2017) consider a multilateral matching market with complements. Alva and Teytelboym (2015) analyze supply chains

showed that any supply chain has a *chain-stable* outcome for which there are no blocking upstream or downstream chains of contracts. Hatfield and Kominers (2012) further showed that, in the presence of network acyclicity, chain-stable outcomes are equivalent to *stable* outcomes i.e. those that are immune to deviations by arbitrary sets of firms. Even under full substitutability, stable/chain-stable outcomes in supply chains may be Pareto inefficient.[3]

While a supply chain may be a good model of production in certain industries (Antràs and Chor, 2013), in general, firms simultaneously supply inputs to *and* buy outputs from other firms (possibly through intermediaries). If this is the case, we say a trading network contains a contract *cycle*. For example, the sectoral input-output network of the U.S. economy, illustrated by Acemoglu et al. (2012, Figure 3), shows that American firms are very interdependent and the trading network contains many cycles. Consider a coal mine that supplies coal to a steel factory. The factory uses coal to produce steel, which is an input for a manufacturing firm that sells mining equipment back to the mine. This creates a contract cycle. However, Hatfield and Kominers (2012) showed that if a trading network without transfers has a contract cycle then stable outcomes may fail to exist. Moreover, Fleiner, Jankó, Schlotter, and Teytelboym (2018) show that checking whether a stable outcome exists—or even whether a given outcome is stable—is computationally intractable.

We show that, even in the presence of contract cycles, outcomes that satisfy a different stability concept—*trail stability*—can still be found. A trail of contracts is a set of contracts which can be ordered in such a way that the buyer in one contract is the seller in the subsequent one. Along a *locally blocking trail*, whenever a firm receives an upstream (downstream) offer, it can either accept it unconditionally or hold the offer and make a myopic, unilateral downstream (upstream) offer. Trail stability rules out any such locally blocking trails. We argue that trail stability is a useful, natural, and intuitive equilibrium concept for the analysis of matching markets in networks because locally blocking trails do not require extensive coordination among recontracting parties. In general trading networks, stable and chain-stable outcomes are trail-stable under full substitutability (but not in general). Trail

in which inputs could be complementary or substitutable with general preferences. Jagadeesan (2017) explores trading networks with complementary inputs in a large market setting. While production decisions often create externalities, we assume that firms only care about the contracts they are involved in (Sasaki and Toda, 1996, Bando, 2012, Pycia, 2012, Pycia and Yenmez, 2015). In addition, we work in a complete information environment; for a treatment with asymmetric information, see Roth (1989), Ehlers and Massó (2007), Chakraborty et al. (2010). Extending our model to incorporate incomplete information and externalities is a promising area for further research.

[3] Inefficiency arises even in two-sided many-to-many matching markets without contracts if agents have multi-unit demands: Blair (1988) and Roth and Sotomayor (1990, p. 177) provide the earliest examples; Echenique and Oviedo (2006), Klaus and Walzl (2009) discuss the setting with contracts. Westkamp (2010) provides necessary and sufficient conditions on the structure contract relationships in the supply chain for chain-stable outcomes to be efficient.

stability is equivalent to chain stability (and therefore to stability under our assumptions) in acyclic trading networks and to pairwise stability in two-sided matching markets.

Trail-stable outcomes correspond to the fixed-points of an operator and form a particular lattice structure for *terminal agents* who can sign only upstream or only downstream contracts. The lattice reflects the classic opposition-of-interests property of two-sided markets, but in our case the opposition of interests is between terminal buyers and terminal sellers. In addition to this strong lattice property, we extend previous results on the existence of buyer- and seller-optimal stable outcomes, the rural hospitals theorem, strategy-proofness as well as comparative statics on firm entry and exit that have only been studied in a supply-chain or two-sided setting under general choice functions.

We then introduce another solution concept called *weak trail stability* which ensures that firms are willing to sign *all* their contracts along a *sequentially blocking trail* (rather than simply upstream-downstream pair along a locally blocking trail). This might be a useful solution concept if the blocking contracts are not fulfilled quickly. Stable and chain-stable outcome are always weakly trail-stable. We show the under full substitutability weakly trail-stable outcomes exist because trail-stable outcomes are weakly trail-stable (without full substitutability, however, trail-stable outcomes may not be weakly trail-stable).

Our work complements an important recent paper by Hatfield et al. (2015) which shows that in general trading networks, under certain conditions, stable outcomes coincide with chain-stable outcomes i.e. those immune to coordinated deviations by a set of firms which is *simultaneously* signing a trail of contracts. Our paper is also related to the stability of (continuous and discrete) network flows discussed by Fleiner (2009, 2014). In these models, agents choose the amount of "flow" they receive from upstream and downstream agents and have preferences over who they receive the "flow" from. The network flow model allows for cycles. However, the choice functions in the network flow models are restricted by Kirchhoff's (current) law (the total amount of incoming (current) flow is equal to the total amount of outgoing flow) and in the discrete case these choice functions are special cases of Ostrovsky (2008). This paper therefore generalizes both of the supply chain and the network flow models, while offering two appealing new solution concepts.

Hatfield et al. (2013) were the first to consider a general trading network model and proved that stable outcomes always exist under full substitutability in a transferable utility (TU) economy. However, TU rules out wealth effects and distortionary frictions, such as sales taxes, bargaining costs, or incomplete financial markets. Fleiner, Jagadeesan, Jankó, and Teytelboym (2018) consider a variation of our model in which every contract specifies a trade and a continuous price (see also Hatfield et al., 2015). They show that, under full substitutability, trail-stable outcomes are essentially equivalent to competitive equilibrium outcomes. In the presence of distortionary frictions, however, they show that competitive equilibria are not stable (in fact, stable outcomes may not exist) and can even be Pareto-comparable.[4] Hence, trail stability can also serve a cooperative interpretation of competitive equilibrium in settings where price-taking behavior is not a plausible assumption.

We proceed as follows. In Section 2, we present the ingredients of the model, including the trading network, main assumptions on choice functions, and terminal agents. In Section 2.5, we introduce stability, chain stability, and trail stability. Then, in Section 3, we report the existence of trail-stable outcomes before digging deeper into their structure. In Section 4, we introduce weak trail stability and describe its relationship to stability and trail stability. Finally, we conclude and outline some directions for future work. Appendix A summarizes all solution concepts, results and nomenclature used in this paper with reference to previous work and describes which previous results we have generalized in our setting of trading networks with general choice functions. Appendix B provides proofs of the main results. In Appendix C, we give sufficient conditions on preferences that ensure that trail-stable and stable outcomes or trail-stable and weakly trail-stable outcomes coincide. Appendix D considers yet another solution concept.

2 Model

2.1 Ingredients

There is finite set of agents (firms or consumers) F and a finite set of contracts (trading network) X.[5] A contract $x \in X$ is a bilateral agreement between a buyer $b(x) \in F$ and a seller $s(x) \in F$. A *(trading) cycle* in X is a sequence of firms $(f_1, \ldots, f_M)$ such that for all $m \in \{1, \ldots, M\}$ there exists a contract x_m such that $s(x_m) = f_m$ and $b(x_m) = f_{m+1}$ (subscripts modulo M). Hence, $F(x) = \{s(x), b(x)\}$ is the set of firms associated with contract x and, more generally, $F(Y)$ is the set of firms associated with contract set $Y \subseteq X$. Denote $X_f^B = \{x \in X | b(x) = f\}$ and $X_f^S = \{x \in X | s(x) = f\}$ the sets of f's upstream and downstream contracts—i.e. the contract for which f is a buyer and a seller, respectively. Clearly, Y_f^B and Y_f^S form a partition over the set of contracts $Y_f = \{y \in Y | f \in F(y)\}$ which involve f, since an agent cannot be a buyer and a seller in the same contract.

[4]Without distortionary frictions, competitive equilibrium outcomes are stable and in the core even in the presence of wealth effects (Fleiner, Jagadeesan, Jankó, and Teytelboym, 2018). In earlier work, Hatfield et al. (2013) and Hatfield and Kominers (2015) made this observation in a TU economy.

[5]The standard justification for this assumption is given by Roth (1984, p. 49): "elements of a [contract] can take on only discrete values; salary cannot be specified more precisely than to the nearest penny, hours to the nearest second, etc." In fact, the finiteness assumption is not necessary for our existence proof. We only require that the set of contracts between any two agents forms a complete lattice.

Our structure is more general than the setting described by Ostrovsky (2008), Westkamp (2010) or Hatfield and Kominers (2012). Each firm $f \in F$ is associated with a vertex of a directed multigraph (F, X) and each contract $x \in X$ is a directed edge of this graph. For any f, the set X_f^B corresponds to the set of incoming edges and X_f^S to the set of outgoing edges. An *acyclic* trading network (a *supply chain*) contains no directed cycles in the graph. Supply chains require a partial order on the firms' positions in the chain although firms may sell to (buy from) any downstream (upstream) level. In our model, we consider general trading networks, which may contain contract cycles (i.e. directed cycles on the graph).

Every firm has a choice function C^f, such that $C^f(Y_f) \subseteq Y_f$ for any $Y_f \subseteq X_f$.[6] We say that choice functions of $f \in F$ satisfy the *irrelevance of rejected contracts (IRC)* condition if for any $Y \subseteq X$ and $C^f(Y) \subseteq Z \subseteq Y$ we have that $C^f(Z) = C^f(Y)$ (Blair, 1988, Alkan, 2002, Fleiner, 2003, Echenique, 2007, Aygün and Sönmez, 2013).[7]

For any $Y \subseteq X$ and $Z \subseteq X$, define the *chosen* set of upstream contracts

$$C_B^f(Y|Z) = C^f(Y_f^B \cup Z_f^S) \cap X_f^B \tag{2.1}$$

which is the set of contracts f chooses as a buyer when f has access to upstream contracts Y and downstream contracts Z. Analogously, define the chosen set of downstream contracts

$$C_S^f(Z|Y) = C^f(Z_f^S \cup Y_f^B) \cap X_f^S \tag{2.2}$$

Hence, we can define *rejected* sets of contracts $R_B^f(Y|Z) = Y_f \setminus C_B^f(Y|Z)$ and $R_S^f(Z|Y) = Z_f \setminus C_S^f(Z|Y)$. An *outcome* $A \subseteq X$ is a set of contracts.

A set of contracts $A \subseteq X$ is *individually rational* for an agent $f \in F$ if $C^f(A_f) = A_f$. We call set A *acceptable* if A is individually rational for all agents $f \in F$. For sets of contracts $W, A \subseteq X$, we say that A is (W, f)-*acceptable* if $A_f \subseteq C^f(W_f \cup A_f)$ i.e. if the agent f chooses all contracts from set A_f whenever she is offered A alongside W. Set of contracts A is W-*acceptable* if A is (W, f)-acceptable for all agents $f \in F$. Note that contract set A is individually rational for agent f if and only if it is $(\emptyset, f)$-acceptable. If $y \in X_f^B$ and $z \in X_f^S$ then $\{y, z\}$ is a (W, f)-*essential pair* if neither $\{y\}$ nor $\{z\}$ is (W, f)-acceptable but $\{y, z\}$ is (W, f)-acceptable. Note that any essential pair consists of exactly one upstream and one downstream contract.

[6] Since firms only care about their own contracts, we can write $C^f(Y)$ to mean $C^f(Y_f)$.

[7] In our setting, IRC is equivalent to the Weak Axiom of Revealed Preference (Aygün and Sönmez, 2013, Alva, 2018).

2.2 Assumptions on choice functions

We can now state our key assumption on choice functions introduced by Ostrovsky (2008) and Hatfield and Kominers (2012).

Definition 1. Choice functions of $f \in F$ are *fully substitutable* if for all $Y' \subseteq Y \subseteq X$ and $Z' \subseteq Z \subseteq X$ they are:

1. *Same-side substitutable* (SSS):
   (a) $R_B^f(Y'|Z) \subseteq R_B^f(Y|Z)$
   (b) $R_S^f(Z'|Y) \subseteq R_S^f(Z|Y)$
2. *Cross-side complementary* (CSC):
   (a) $R_B^f(Y|Z) \subseteq R_B^f(Y|Z')$
   (b) $R_S^f(Z|Y) \subseteq R_S^f(Z|Y')$

Contracts are fully substitutable if every firm regards any of its upstream or any of its downstream contracts as substitutes, but its upstream and downstream contracts as complements. Hence, rejected downstream (upstream) contracts continue to be rejected whenever the set of offered downstream (upstream) contracts expands or whenever the set of offered upstream (downstream) contracts shrinks.

2.3 Laws of Aggregate Demand and Supply

We first re-state the familiar Laws of Aggregate Demand and Supply (LAD/LAS) (Hatfield and Milgrom, 2005, Hatfield and Kominers, 2012). LAD (LAS) states that when a firm has more upstream (downstream) contracts available (holding the same downstream (upstream) contracts), the number of downstream (upstream) contracts the firms chooses does not increase more than the number of upstream (downstream) contracts the firm chooses. Intuitively, an increase in the availability of contracts on one side, does not increase the difference between the number of contracts signed on either side.[8]

Definition 2. Choice functions of $f \in F$ satisfy the *Law of Aggregate Demand* if for all $Y, Z \subseteq X$ and $Y' \subseteq Y$

$$|C_B^f(Y|Z)| - |C_B^f(Y'|Z)| \geq |C_S^f(Z|Y)| - |C_S^f(Z|Y')|$$

and the *Law of Aggregate Supply* if for all $Y, Z \subseteq X$ and $Z' \subseteq Z$

$$|C_S^f(Z|Y)| - |C_S^f(Z'|Y)| \geq |C_B^f(Y|Z)| - |C_B^f(Y|Z')|$$

We can easily show that LAD/LAS imply IRC, extending the result by Aygün and Sönmez (2013).

[8] This is an extension of the "cardinal monotonicity" condition (Alkan, 2002).

Lemma 1. Suppose that choice function of $f \in F$ satisfies full substitutability and LAD/LAS. Then the choice function of f satisfies IRC.

Proof of Lemma 1. Consider $Y \subset X_f$ and $z \in X_f^B \setminus Y$ such that $z \notin C^f(Y \cup \{z\})$. Then (1), from SSS, we have that $C_B^f(Y \cup \{z\}) \subseteq C_B^f(Y)$, and (2), from CSC, we have that $C_S^f(Y \cup \{z\}) \supseteq C_S^f(Y)$. If choice functions satisfy LAD/LAS, then we have that $|C_B^f(Y)| - |C_S^f(Y))| \leq |C_B^f(Y \cup \{z\})| - |C_S^f(Y \cup \{z\})|$. But then both of the above the set inclusions (1) and (2) must hold with equality, so $C^f(Y \cup \{z\}) = C^f(Y)$ as required. □

2.4 Terminal agents and terminal superiority

We now introduce some terminology that describes contracts of agents, who only act as buyers or only act as sellers. A firm f is a *terminal seller* if there are no upstream contracts for f in the network and f is a *terminal buyer* if the network does not contain any downstream contracts for f. An agent who is either a terminal buyer or a terminal seller is called a *terminal agent*. Let $\mathcal{T}$ denote the set of terminal agents in F and for a set A of contracts let us denote the *terminal contracts of A* by $A_{\mathcal{T}} = \bigcup\{A_f | f \in \mathcal{T}\}$. A set Y of contracts is *terminal-acceptable* if there is an acceptable set A of contracts such that $Y = A_{\mathcal{T}}$. If A and W are terminal-acceptable sets of contracts then we say that A is *seller-superior* to W (denoted by $A \succeq^S W$) if $C_f(A_f \cup W_f) = A_f$ for each terminal seller f and $C_g(A_g \cup W_g) = W_g$ for each terminal buyer g. Similarly, A is *buyer-superior* to W (denoted by $A \succeq^B W$) if $C_f(A_f \cup W_f) = W_f$ for each terminal seller f and $C_g(A_g \cup W_g) = A_g$ for each terminal buyer g.[9] Clearly, these relations are opposite, that is, $W \succeq^S A$ if and only if $A \succeq^B W$ holds. Whenever either relation holds, we call this partial order on outcomes *terminal superiority*. Terminal agents are going to play a key role when we describe the structure of outcomes in trading networks.

2.5 Solution concepts

We start off by defining two solution concepts that have appeared in previous work.

Definition 3 (Hatfield and Kominers 2012)**.** An outcome $A \subseteq X$ is *stable*[10]

[9] The superiority partial order is equivalent to the revealed preference relation (Alkan and Gale, 2003, Chambers and Yenmez, 2017).

[10] Klaus and Walzl (2009) consider "weak setwise stable" outcomes which are immune to blocks in which sets of agents must also agree on which contracts they can drop. Westkamp (2010) considers "group-stable" outcomes which are immune to blocks in which sets of agents can sign better (rather the best) contracts (also known as "setwise stable outcomes", see Sotomayor, 1999, Echenique and Oviedo, 2006, Klaus and Walzl, 2009).

1. A is acceptable.

2. There exist no non-empty blocking set of contracts $Z \subseteq X$, such that $Z \cap A = \emptyset$ and Z is (A, f)-acceptable for all $f \in F(Z)$.

Stable outcomes are immune to deviations by sets of firms, which can re-contract freely among themselves while keeping any of their existing contracts. Stable outcomes always exist in acyclic networks if choice functions are fully substitutable. In order to study more general trading networks, we first introduce trails of contracts.

Definition 4. A non-empty set of contracts T is a *trail* if its elements can be arranged in some order $(x_1, \ldots, x_M)$ such that $b(x_m) = s(x_{m+1})$ holds for all $m \in \{1, \ldots, M-1\}$ where $M = |T|$.

While a trail may not contain the same contract more than once, it may include the same agents any number of times. For example, in Figure 1, there is a trail $\{w, z, y\}$ that starts from firm m (or j), ends at firm j (or m), and "visits" firm k.

Definition 5 (Hatfield et al. 2015)**.** An outcome $A \subseteq X$ is *chain-stable* if

1. A is acceptable.

2. There is no trail T, such that $T \cap A = \emptyset$ and T is (A, f)-acceptable for all $f \in F(T)$.

Hatfield et al. (2015) show that in general trading networks stable outcomes are equivalent to chain-stable outcomes whenever choice functions satisfy full substitutability and Laws of Aggregate Demand and Supply.[11] However, Fleiner (2009) and Hatfield and Kominers (2012) show that stable outcomes may not exist in general trading networks (see also Example 1 below). Moreover, Fleiner, Jankó, Schlotter, and Teytelboym (2018) show that both problems of determining whether a stable outcome exists and determining whether an outcome is stable are computationally intractable.

The non-existence and computational intractability of stable outcomes motivates us to define an alternative solution concept. We first define *trail stability*, which coincides with pairwise stability in a two-sided many-to-many matching market with contracts and with chain stability in supply chains.

Definition 6. An outcome $A \subseteq X$ is *trail-stable* if

1. A is acceptable.

[11] Hatfield et al. (2015) refer to trails as "chains" hence the name "chain stability". This extends the definition of chain stability introduced by Ostrovsky (2008).

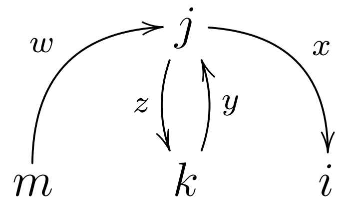

Figure 1: Trading network in Examples 1, 3, and 4.

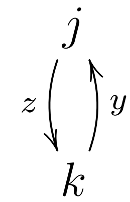

Figure 2: Trading network in Example 2.

2. There is no trail $T = \{x_1, x_2, \ldots, x_M\}$, such that $T \cap A = \emptyset$ and

   (a) $\{x_1\}$ is (A, f_1)-acceptable for $f_1 = s(x_1)$, and

   (b) $\{x_{m-1}, x_m\}$ is (A, f_m)-acceptable for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq M$ and

   (c) $\{x_M\}$ is (A, f_{M+1})-acceptable for $f_{M+1} = b(x_M)$.

   The above trail T is called a *locally blocking trail to A*.

Trail stability is a natural solution concept when firms interact mainly with their buyers and suppliers and deviations by arbitrary sets of firms are difficult to arrange. In a trail-stable outcome, no agent wants to drop his contracts and there exists no sequence of *consecutive* bilateral contracts comprising a trail preferred by all the agents in the trail to the current outcome. First, f_1 makes an unilateral offer of x_1 (the first contract in the trail) to the buyer f_2. The buyer f_2 then either unconditionally accepts the offer (forming a locally blocking trail) or conditionally accepts the seller's offer while looking to make a contract offer (x_2) to another buyer f_3. If f_2's buyer in x_2 happens to be f_1, then f_1 considers the offer of x_2 on it own. If f_1 accepts, we have a locally blocking trail. If f_2's buyer is not f_1, then his buyer f_3 either accepts x_2 unconditionally or looks for another seller f_4 after a conditional acceptance of x_2. The trail of these linked conditional contract offers continues until the last buyer f_{M+1} in the trail unconditionally accepts the upstream contract offer x_M.[12]. Note that all intermediate agents only need to myopically decide whether they want to choose pairs of upstream-downstream contract that "pass though" the agents along the trail. In other words, whenever an agent receives a contract offer, he is "activated" to either accept or reject it or to hold this offer while making his own contract offer.

The following example illustrates that trail-stable outcomes are not necessarily stable.[13]

[12]The trail and the order of conditional acceptances can, of course, be reversed with f_{M+1} offering the first upstream contract to seller f_M and so on.

[13]Example 1 is similar to examples in Fleiner (2009, p. 12) and Hatfield and Kominers (2012, Fig. 3, p. 13).

Example 1 (Trail-stable outcomes are not necessarily stable)**.** Consider four contracts x, y, z and w. Assume that $i = b(x)$, $j = s(x) = s(z) = b(y) = b(w)$, $k = b(z) = s(y)$ and $m = s(w)$ (see Figure 1). Agents have the following preferences that induce fully substitutable choice functions:[14]

$$\begin{aligned}
\succ_i &: \{x\} \succ_i \emptyset \\
\succ_m &: \{w\} \succ_m \emptyset \\
\succ_j &: \{x, y, w\} \succ_j \{z, y, w\} \succ_j \{x, y\} \succ_j \{z, y\} \succ_j \{w\} \succ_j \emptyset \\
\succ_k &: \{z, y\} \succ_k \emptyset
\end{aligned}$$

and other outcomes are not acceptable. A trail-stable outcome exists: $A = \{w\}$. The trail-stable outcome $\{w\}$ is Pareto-inefficient as $\{z, y, w\}$ makes j and k better off without making i and m worse off. There is, however, no stable outcome.[15]

To illustrate trail stability further, let us drop agents i and m and their corresponding contracts from the example above (Figure 2). The new preferences of j are $\{y, z\} \succ_j \emptyset$. There is one stable outcome $\{y, z\}$. There are, however, two weakly trail-stable outcomes: $\emptyset$ and $\{y, z\}$. Is $\emptyset$ a reasonable possible outcome of this market? If coordination is difficult, due to bargaining costs for example, firms may find it difficult to pin down blocking sets, especially in large markets. Trail stability therefore provides a natural solution concept for matching markets in which firms have a limited ability to coordinate their decisions in the trading network.

3 Existence and properties of trail-stable outcomes

We can now state the first key result of this paper.

Theorem 1. Suppose that choice functions satisfy full substitutability and IRC. Then there exists a trail-stable outcome.[16]

This theorem establishes a positive existence result for an appealing solution concept in general trading networks: under the usual assumption of full substitutability, trail-stable outcomes always exist.

[14] In all our examples, $\succ$ denotes a strict preference relation. Choice functions induced by strict preferences satisfy IRC.

[15] One can check that there is no stable outcome since $\{w\} \succ_j \{x, w\} \succ_k \{x, z, w\} \succ_{i,j} \{z, y, w\} \succ_{j,k} \{w\}$ and other outcomes are not acceptable.

[16] Our results do not contradict Theorem 5 on the non-existence of stable outcomes in Hatfield and Kominers (2012) since Theorem 1 only considers the existence of trail-stable outcomes.

In order to prove Theorem 1, we use tools familiar to matching theory, such as the Tarski fixed-point theorem (Adachi, 2000, Fleiner, 2003, Echenique and Oviedo, 2006, Hatfield and Milgrom, 2005, Ostrovsky, 2008, Hatfield and Kominers, 2012). Let X^B and X^S be two subsets of X which represent the set of contracts offered to buyers and sellers. We define an isotone operator Φ that acts on (X^B, X^S) and show that any fixed-point $(\dot{X}^B, \dot{X}^S)$ of Φ corresponds to a trail-stable outcome $A = \dot{X}^B \cap \dot{X}^S$. These tools allow us to explore properties of trail-stable outcomes that have previously only been explored in a supply-chain or a two-sided setting.

As we have already seen, trail-stable outcomes are not always stable or chain-stable even under full substitutability. However, the converse is true.

Proposition 1. Suppose that choice functions satisfy full substitutability and IRC. Then any stable/chain-stable outcome is trail-stable.

Full substitutability is key to this result as the following example shows.

Example 2 (Stability and chain stability do not imply trail stability without full substitutability)**.** Consider two contracts y, z. Assume that $j = s(z) = b(y)$, $k = b(z) = s(y)$ (see Figure 2). Agents have the following preferences:

$$\succ_j : \{z\} \succ_j \{y\} \succ_j \emptyset$$
$$\succ_k : \{z, y\} \succ_k \emptyset.$$

and other outcomes are not acceptable. Note that k's preference are fully substitutable, but j's preferences are not. The empty set of contracts is stable (and chain-stable), however, it is not trail-stable since $\{z, y\}$ is a locally blocking trail.

This example highlights that locally blocking trails need not be acceptable (alongside other contracts) themselves: firm j "forgets" that it offered contract z when it considers the terminal contract y offered in return by k. In Section 4, we will consider a solution concept which ensures that agents are willing to accept all the contracts along a locally blocking trail (perhaps alongside other contracts).

3.1 Structure of trail-stable outcomes

Recall that in the marriage model of Gale and Shapley, the existence of man-optimal and woman-optimal stable matchings follow from the well-known lattice structure of stable matchings. The key to extending this result to trading networks is to consider only terminal agents. We say that a trail-stable outcome A_{max} (A_{min}) that is *buyer-optimal* (*seller-optimal*) if any terminal buyer (terminal seller) prefers it to any other outcome i.e. for any trail-stable $Z \subseteq X$, we have that $C^f(A_{max} \cup Z) = A_{f,max}$.

Proposition 2. Suppose that choice functions satisfy full substitutability and IRC. Then the set of trail-stable outcomes contains buyer-optimal and seller-optimal outcomes.

Proposition 2 extends Theorem 2 by Ostrovsky (2008) and Theorem 4 by Hatfield and Kominers (2012), which establish the existence of buyer- and seller-optimal outcomes in acyclic trading networks.[17] We say that $Y \subseteq X$ is *terminal-trail-stable* if there is a trail-stable outcome $A \subseteq X$ such that $Y = A_{\mathcal{T}}$.

Proposition 3. Suppose that choice functions satisfy full substitutability and LAD/LAS. Then the terminal-trail-stable contract sets form a lattice under terminal superiority.

Proposition 3 shows that whenever LAD/LAS holds choice functions of terminal agents define a natural partial order on outcomes and the terminal-trail-stable contract sets form a lattice under this order.[18] Note that for the lattice and the opposition-of-interests structure, only terminal agents play a role: two outcomes are equivalent if all the terminal agents have the same set of contracts. Indeed, if A^1 and A^2 are trail-stable outcomes then there is a trail-stable outcome A^+ such that all terminal buyers prefer A^+ to both A^1 and A^2 and all sellers prefer any of A^1 and A^2 to A^+.[19] This establishes full "polarization of interests" in trail-stable outcomes in the sense of (Roth, 1985) and immediately implies the existence of buyer-optimal (A_{max}) and seller-optimal (A_{min}) trail-stable outcomes. Therefore, our result substantially strengthens and generalizes the previous results by Roth (1985), Blair (1988), Echenique and Oviedo (2006) and Hatfield and Kominers (2012).

The lattice structure of fully-trail stable outcomes allows us to straightforwardly extend two well-known properties of stable outcomes that have been known in two-sided matching markets and acyclic trading networks. One such property is the classic "rural hospitals theorem", which shows that in every stable allocation of a two-sided many-to-one doctor-hospital matching market, the same number of doctors are matched to every hospital (Roth, 1986). In buyer-seller networks, we can instead consider the difference between the number of upstream and downstream contracts that firms sign (Hatfield and Kominers, 2012). The following proposition gives the most general rural hospital theorem result.

[17] This is a common property of stable outcomes in two-sided markets with substitutable choice functions, however, it typically fails in richer matching models (Pycia and Yenmez, 2015, Alva, 2015, Alva and Teytelboym, 2015).

[18] In fact, our proof shows that the terminal-trail-stable contract sets form a *sublattice* (Blair, 1988, Fleiner, 2003, Echenique and Oviedo, 2006).

[19] Of course, the same holds for if we exchange the role of buyers and sellers.

Proposition 4. Suppose that choice functions satisfy full substitutability and LAD/LAS. Then, for each firm, the difference between the number of upstream contracts and the number of downstream contracts is invariant across trail-stable outcomes.[20]

The lattice structure of trail-stable outcomes also gives a (somewhat weak) mechanism design result.[21] A mechanism $\mathcal{M}$ is a mapping from a profile of agents' choice functions, $\mathbf{C}^F = (C^f)_{f \in F}$, to the set of outcomes.

Definition 7. A mechanism is *group strategy-proof* for $G \subseteq F$ if for any $\bar{G} \subseteq G$, there does not exist a choice function profile $\bar{\mathbf{C}}^{\bar{G}}$ such that for outcomes $\bar{A} = \mathcal{M}(\bar{\mathbf{C}}^{\bar{G}}, \mathbf{C}^{F \setminus \bar{G}})$ and $A = \mathcal{M}(\mathbf{C}^F)$ we have that $C^f(\bar{A} \cup A) = \bar{A}$ for every $f \in \bar{G}$.

A mechanism is group strategy-proof for a group of agents if they cannot jointly manipulate their choice functions and obtain an outcome that is better for all of them. Like Hatfield and Kominers (2012), we are only going to consider group strategy-proofness for terminal agents. We generalize their Theorem 10 with the following result.

Proposition 5. Suppose that choice functions satisfy full substitutability and LAD/LAS and, additionally, all terminal buyers (terminal sellers) demand at most one contract. Then any mechanism that selects the buyer-optimal (seller-optimal) trail-stable outcome is group strategy-proof for all terminal buyers.

As is well known, the assumptions that underpin Proposition 5—unit demands and extreme one-sidedness—cannot be substantially relaxed (Hatfield and Kojima, 2009).

3.2 Trail-stable outcomes and comparative statics

The second set of properties of trail-stable outcomes concerns the effect of entry and exit of new firms in the trading network. This type of comparative static analysis is well-studied in two-sided matching markets (Gale and Sotomayor, 1985, Crawford, 1991, Blum et al., 1997, Hatfield and Milgrom, 2005). More recently, Ostrovsky (2008) and Hatfield and Kominers (2013) extended these results the case of supply chains.

First, let us consider what happens when a terminal seller is added to the market. More formally, let $F' = F \cup \{f'\}$ and let A'_{max} and A'_{min} be the buyer-optimal and the seller-optimal trail-stable outcomes in F' respectively.

[20] Theorem 4 in Fleiner (2014), which states that any two stable flows agree on terminal contracts and any two stable flows have the same number of assignments, is a further strengthening of Propositions 3 and Propositions 4 in the special case of network flows.

[21] One design application of trading networks is a peer-to-peer electricity market in which many consumers also generate electricity (Morstyn et al., 2018).

Proposition 6. Suppose that choice functions satisfy full substitutability and IRC. Suppose moreover that a new terminal seller f' whose choice function is fully substitutable and satisfies IRC enters the market. Then

- every terminal seller $f \neq f'$ prefers A_{max} to A'_{max} and prefers A_{min} to A'_{min}, and
- every terminal buyer f prefers A'_{max} to A_{max} and prefers A'_{min} to A_{min}.[22]

Proposition 6 says that with a new seller, the seller-optimal outcome A_{min} and the buyer-optimal outcome A_{max} move in the direction favorable to terminal buyers and unfavorable to terminal sellers. Symmetrically, when a terminal buyer is added or if a seller leaves, A_{min} and A_{max} move in the opposite direction. In other words, more competition on one end of an industry is bad for the agents on that end and good for the agents on the other end.[23] Proposition 6 generalizes Theorem 3 in Ostrovsky (2008).

Now consider the following *market readjustment process*: When the new terminal seller f' enters, and we already have a trail-stable outcome A with corresponding fixed point $(\dot{X}^B, \dot{X}^S)$ then let X be the set of all contracts in the new network, and let us define $(\dot{X}'^B, \dot{X}'^S) = (\dot{X}^B, \dot{X}^S \cup X_{f'})$. Operator Φ' acts on (X'^B, X'^S) using choice functions of F'. Let $(\hat{X}^B, \hat{X}^S)$ be the fixed point of the iteration of fuction Φ, with associated outcome $\hat{A} = \hat{X}^B \cap \hat{X}^S$ which is the result of the market readjustment process.

Proposition 7. Suppose that choice functions satisfy full substitutability and IRC. Consider a trail-stable outcome A with associated buyer and seller offer sets X^B and X^S. Suppose a new terminal seller f' whose choice function is fully substitutable and satisfies IRC enters the market and let $\hat{A}$ be the result of the market readjustment process. Then, all terminal sellers prefer A to $\hat{A}$ and all terminal buyers (other than f') prefer $\hat{A}$ to A.[24]

An analogous result is obtained when terminal buyers and terminal sellers exit the market so Proposition 7 generalizes the Theorem in Hatfield and Kominers (2013).

[22] The opposite holds when f' is a terminal buyer.

[23] It is also possible to prove an analogous result to Proposition 6 when choice function of terminal agents expands under IRC and full substitutability. Choice function of a terminal agent $\widehat{C}^f$ on 2^X is an expansion of choice function C^f on 2^X if, for every $Y \subseteq X$, $\widehat{C}^f(Y) \subseteq C^f(Y)$ (Echenique and Yenmez, 2015, Chambers and Yenmez, 2017). Then suppose all choice functions satisfy full substitutability and IRC and the choice function of one of the terminal buyers (sellers) expands. Then every terminal seller (buyer) prefers the new buyer-optimal and seller-optimal trail-stable outcomes to the old ones and vice versa for the buyers (sellers). This is a straightforward adaptation of Theorem 2 in (Chambers and Yenmez, 2017).

[24] The opposite holds when f' is a terminal buyer.

4 Weak trail stability

As we saw in the previous section, the entire locally blocking trail does not need to be acceptable for agents who are participating in the block. If contracts are not fulfilled immediately, then it might be important to ensure that agents want to select all the contracts along a blocking trail. Let us consider a trail $T = \{x_1, ..., x_M\}$ whose elements are arranged in a sequence $(x_1, ..., x_M)$ and define $T_f^{\leq m} = \{x_1, ..., x_m\} \cap T_f$ to be firm f's contracts out of the first m contracts in the trail and $T_f^{\geq m} = \{x_m, ..., x_M\} \cap T_f$ to be firm f's contracts out of the last $M - m + 1$ contracts in the trail (where $m \in \{1, \ldots, M\}$).

Definition 8. An outcome $A \subseteq X$ is *weakly trail-stable* if

1. A is acceptable.
2. There is no trail $T = \{x_1, x_2, \ldots, x_M\}$, such that $T \cap A = \emptyset$ and
   (a) $\{x_1\}$ is (A, f_1)-acceptable for $f_1 = s(x_1)$ and
   (b) At least one of the following two options holds:
      i. $T_{f_m}^{\leq m}$ is (A, f_m)-acceptable for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq M$, or
      ii. $T_{f_m}^{\geq m-1}$ is (A, f_m)-acceptable for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq M$
   (c) $\{x_M\}$ is (A, f_{M+1})-acceptable for $f_{M+1} = b(x_M)$.

   The above trail T is called a *sequentially blocking trail to* A.

The agents who are participating in a sequentially blocking trail need to choose all the contracts in the trail whenever the trail "loops back" to them. As the sequentially blocking trail grows, we ensure that each intermediate agent wants to choose *all* his contracts along the trail. This ensures that the sequentially blocking trail is as a whole is selected by all agents in the block. Therefore, weak trail stability might be a more suitable solution concept for cases where contracts last longer or as a long-run prediction of outcomes. The next result is immediate so we state it without proof.

Proposition 8. Suppose that choice functions satisfy IRC. Then any stable/chain-stable outcome is weakly trail-stable.

The full substitutability assumption is not required for Proposition 8, but, of course, the existence of stable or chain-stable outcome is not guaranteed in trading networks even under full substitutability. On the other hand, without full substitutability, trail-stable outcomes may not be weakly trail-stable as the following example shows.

Example 3 (Trail stability does not imply weak trail stability without full substitutability)**.** Consider agents and contracts described in Example 1 and Figure 1. Agents have the following preferences:

$$\begin{aligned}\succ_m &: \{w\} \succ_m \emptyset \\ \succ_i &: \{x\} \succ_i \emptyset \\ \succ_k &: \{z, y\} \succ_k \emptyset \\ \succ_j &: \{w, x, z, y\} \succ_j \{w, z\} \succ_j \emptyset.\end{aligned}$$

and other outcomes are not acceptable. The preferences of all agents, except j, are fully substitutable. The empty set is a trail-stable outcome, but it is not weakly trail-stable since $\{w, z, x, y\}$ is a sequentially blocking trail when m makes the first offer.

However, under full substitutability trail-stable outcomes are always weakly trail-stable.

Proposition 9. Suppose that choice functions satisfy full substitutability and IRC. Then any trail-stable outcome is weakly trail-stable.

The existence of weakly trail-stable outcomes under full substitutability is therefore an immediate consequence of Theorem 1 and Proposition 9.

Corollary 1. Suppose that choice functions satisfy full substitutability and IRC. Then there exists a weakly trail-stable outcome.

One may wonder whether under full substitutability trail-stable and weakly trail-stable outcomes in fact coincide. This is not the case—the converse of Proposition 9 is false as the next example shows.

Example 4 (Weakly trail-stable outcomes are not always trail-stable even under full substitutability)**.** Consider agents and contracts described in Example 1 and Figure 1. Agents have the following preferences that induce fully substitutable choice functions:

$$\begin{aligned}\succ_m &: \{w\} \succ_m \emptyset \\ \succ_i &: \{x\} \succ_i \emptyset \\ \succ_k &: \{z, y\} \succ_k \emptyset \\ \succ_j &: \{z, y\} \succ_j \{w, z\} \succ_j \{y, x\} \succ_j \emptyset.\end{aligned}$$

and other outcomes are not acceptable. For outcome $A = \emptyset$, the trail $\{w, z, y, x\}$ is locally blocking trail but not trail-blocking. Therefore, weakly trail-stable outcomes are $\emptyset$ and $\{z, y\}$ but the only trail-stable outcome is $\{z, y\}$.

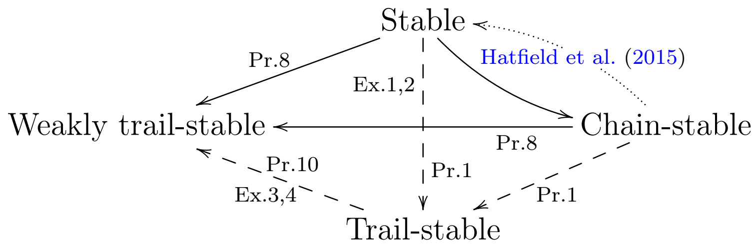

Figure 3: Relationships between solution concepts in general trading networks. Solid line: holds under IRC. Dashed line: holds under full substitutability and IRC. Dotted line: holds under full substitutability, IRC, and LAD/LAS. Arrows show which propositions establish the relationship and which examples examine the assumptions or the converse.

Therefore, under full substitutability, trail stability is a refinement of weak trail stability.

Figure 3 summarizes the relationships between various solution concepts in general trading networks that we have established in this paper. Stable and chain-stable outcomes may not exist even under full substitutability and they are not equivalent without the additional LAD/LAS assumption (see Example 1 in Hatfield et al., 2015).

5 Conclusion

Stability is an appealing solution concept, but in general trading networks stable outcomes may not exist. In this paper, we introduced a new solution concept for general trading networks, called trail stability. We showed that any trading network has a trail-stable outcome when choice functions are fully substitutable. Indeed, full substitutability is crucial for existence of trail-stable outcomes since previous maximal domain results for many-to-many matching markets apply in our case (see, for example, Hatfield and Kominers (2012, Theorem 6) and Hatfield and Kominers (2017, Theorem 2)).[25] Trail-stable outcomes have a natural lattice structure and inherit a host of properties studied in two-sided and supply-chain settings. We also considered an alternative solution concept—weak trail stability—which is implied by trail stability under full substitutability. Trail stability is a attractive solution concept for trading networks: in a version of our model with continuous prices, Fleiner, Jagadeesan, Jankó, and Teytelboym (2018) show that competitive equilibrium outcomes are trail-stable and under full substitutability essentially any trail-stable outcome can be supported by competitive equilibrium prices.

There are at least four fruitful areas for further research. The first would be to examine

[25]When firms have quasilinear utility functions, (full) substitutability is not necessary for competitive equilibrium and even when all agents have complementary preferences competitive equilibrium may exist (Baldwin and Klemperer, 2015, Drexl, 2013, Hatfield and Kominers, 2015, Teytelboym, 2014).

the structure of weakly trail-stable outcomes. Second, one might look for weaker sufficient conditions on preferences to establish the coincidence between weakly trail-stable, trail-stable, stable outcomes (Appendix C). Third, it would be interesting to find more market design applications of our model (Morstyn et al., 2018). Finally, it would be useful to understand how our model can be tested and estimated empirically (Fox, 2017).

A Relationship to previous work

	General networks	General choice functions	Existence and structure	New solution concepts used
Ostrovsky (2008)	✗ acyclic	✓	✓	Chain-stable
Westkamp (2010)	✗ acyclic	✓	✓	Group-stable, Core
Hatfield and Kominers (2012)	✗ acyclic	✓	✓	Stable
Hatfield et al. (2013), Hatfield and Kominers (2015)	✓	✗ quasilinear (TU)	✓	Strong group-stable
Hatfield et al. (2015)	✓	✓	✗	Chain-stable
This paper	✓	✓	✓	Trail-stable Weakly trail-stable

Figure 4: Relationship to previous work.

Paper	Theorem	Description	Generalization in this paper
Ostrovsky (2008)	Theorem 1	Existence of stable outcomes	Theorem 1 and Proposition 1
Ostrovsky (2008); Hatfield and Kominers (2012)	Theorem 2; Theorem 4	Buyer- and seller-optimality	Proposition 2 and Proposition 3
Hatfield and Kominers (2012)	Theorem 8	Rural hospitals theorem	Proposition 4
Hatfield and Kominers (2012)	Theorem 10	Strategy-proofness	Proposition 5
Ostrovsky (2008)	Theorem 3	Firm entry	Proposition 6
Hatfield and Kominers (2013)	Theorem	Vacancy chain dynamics	Proposition 7

Figure 5: Previous results generalized in this paper to a trading network setting with general choice functions.

B Proofs

We first prove Theorem 1 on the existence of trail-stable outcomes. We then prove Propositions 2 and 3—these are the most technically challenging results.

The lattice structure of trail-stable outcomes immediately gives the rural hospitals theorem and the strategyproofness result for terminal agents (Propositions 4 and 5). Then we prove Propositions 6 and 7 which examine comparative statics of trail-stable outcomes.

Finally, we prove Propositions 1, 9, 10, and 11 which describe the relationships between stable, trail-stable, and weakly trail-stable outcomes (proof of Proposition 8 is immediate).

Note that we sometimes refer to singleton sets of contracts $\{x\}$ as "contract x" to avoid saying "a set containing contract x".

B.1 Proof of Theorem 1

Consider Y^B and Z^S, which are subsets of X, and represent sets of available upstream and downstream contracts for all agents, respectively. Define a lattice L with the ground set $X \times X$ with an order $\sqsubseteq$ such that $(Y^B, Z^S) \sqsubseteq (Y'^B, Z'^S)$ if $Y^B \subseteq Y'^B$ and $Z^S \supseteq Z'^S$.

Furthermore, define a mapping Φ as follows:

$$\begin{aligned} \Phi_B(Y^B, Z^S) &= X \setminus R_S(Z^S | Y^B) \\ \Phi_S(Y^B, Z^S) &= X \setminus R_B(Y^B | Z^S) \\ \Phi(Y^B, Z^S) &= (\Phi_B(Y^B, Z^S), \Phi_S(Y^B, Z^S)) \end{aligned}$$

where $R_S(Z^S|Y^B) = \bigcup_{f \in F} R_S^f(Z^S|Y^B)$ and $R_B(Y^B|Z^S) = \bigcup_{f \in F} R_B^f(Y^B|Z^S)$. Clearly, Φ is isotone (Fleiner, 2003, Ostrovsky, 2008, Hatfield and Kominers, 2012) on L. We rely on the following well-known fixed point theorem of Tarski.

Theorem B.1. (Tarski, 1955) Let L be a complete lattice and let $\Phi : L \to L$ be an isotone mapping. Then the set of fixed points of Φ in L is also a complete lattice.

Subsequent to the circulation of the first draft of this paper, Adachi (2017) gave an alternative proof of this Theorem 1 using the T-operator defined by Ostrovsky (2008).

Proof of Theorem 1. Existence of fixed-points of Φ follows from Theorem B.1 since $(X \times X, \sqsubseteq)$ is a complete lattice.[26]

We claim that every fixed point $(\dot{X}^B, \dot{X}^S)$ of Φ corresponds to an outcome $\dot{X}^B \cap \dot{X}^S = A$ that is trail-stable. First, we show that A is acceptable. We claim that if $(\dot{X}^B, \dot{X}^S)$ is a fixed point then $\dot{X}^S \cup \dot{X}^B = X$. To see this suppose for contradiction that there is a contract $x \notin \dot{X}^S \cup \dot{X}^B$. Then $x \notin R_S(\dot{X}^S|\dot{X}^B)$ therefore $x \in X \setminus R_S(\dot{X}^S|\dot{X}^B) = \dot{X}^B$. So it must be that $x \in \dot{X}^S \cup \dot{X}^B$. This implies that $R_S(\dot{X}^S|\dot{X}^B) = X \setminus \dot{X}^B = \dot{X}^S \setminus A$ so $C_S(\dot{X}^S|\dot{X}^B) = A$ and similarly $C_B(\dot{X}^B|\dot{X}^S) = A$. From this, we can see that A is acceptable.

Second, we show that A is trail-stable. This is similar to Step 1 of the Proof of Lemma 1 in Ostrovsky (2008). Suppose that $T = \{x_1, \ldots, x_M\}$ is a locally blocking trail and assume towards a contradiction that $T \cap A = \emptyset$. Since we have that $x_1 \in C_S^{s(x_1)}(A \cup \{x_1\}|A)$,

[26] Hence, we do not actually require the assumption of the finiteness of contracts as long as lattice L is appropriately defined. However, we maintain this assumption for ease of comparison with previous results.

we must have that $x_1 \in C_S^{s(x_1)}(\dot{X}^S \cup \{x_1\}|A)$. Since if $C_S^{s(x_1)}(\dot{X}^S \cup \{x_1\}|A) \subseteq \dot{X}^S$ then by IRC $C_S^{s(x_1)}(\dot{X}^S \cup \{x_1\}|A) = A$, therefore $C_S^{s(x_1)}(A \cup \{x_1\}|A) = A$. We also have that $x_1 \in C_S^{s(x_1)}(\dot{X}^S \cup \{x_1\}|\dot{X}^B)$ by CSC. If $x_1 \in \dot{X}^S$, then $x_1 \in \dot{X}^B = X \setminus R_S(\dot{X}^S|\dot{X}^B)$. But we assumed that $x_1 \notin A$, so $x_1 \in \dot{X}^B$.

Now, consider x_2. By definition of a locally blocking trail, we have that $x_2 \in C_S^{s(x_2)}(A \cup \{x_2\}|A \cup \{x_1\})$. Once again by full substitutability and IRC, we obtain that and $x_2 \in C_S^{s(x_2)}(\dot{X}^S \cup \{x_2\}|\dot{X}^B \cup \{x_1\})$. If $x_2 \in \dot{X}^S$, then $x_2 \in \dot{X}^B = X \setminus R_S(\dot{X}^S|\dot{X}^B)$. But we assumed that $x_2 \notin A$, so $x_2 \in \dot{X}^B$. Now proceed by induction, we show that every $x \in T$ is in $\dot{X}^B$. Consider the last contract x_M. Since $x_m \in C_B^{b(x_M)}(A \cup x_M|A)$, using the same argument we had for x_1, we get that $x_M \in \dot{X}^S$. A contradiction.

Now we show that every trail-stable outcome corresponds to a fixed point.

Suppose A is trail-stable. For every $x_i \notin A$, if there exists a trail $\{x_1, x_2, \dots, x_i\}$ such that

- $\{x_1\}$ is $(A, s(x_1))$-acceptable, and
- $\{x_{m-1}, x_m\}$ is (A, f_m)-acceptable for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq i$,

then let $x_i \in X_0^B$. Otherwise, let $x_i \in X_0^S$. Let $\dot{X}^B = A \cup X_0^B$ and $\dot{X}^S = A \cup X_0^S$. Clearly $\dot{X}^S \cup \dot{X}^B = X$.

Outcome A is acceptable, so $C^f(A) = A_f$ for all $f \in F$. For every firm f, if $f = s(x)$ and $x \in \dot{X}^S \setminus A$ then $x \notin C^f(A \cup \{x\})$ otherwise x would be in $\dot{X}^B$. By SSS, we have that $C_S^f(\dot{X}^S|A) = A$. And if $f = b(y)$ and $y \in \dot{X}^B \setminus A$ then $y \notin C^f(A \cup \{y\})$ otherwise the trail ending in y would be a locally blocking trail. By SSS, we then also have that $C_B^f(\dot{X}^B|A) = A$. Moreover, $\{x, y\} \nsubseteq C^f(A \cup \{x, y\})$ otherwise x would be in $\dot{X}^B$. Putting these statements together, we have that $C_S(\dot{X}^S|\dot{X}^B) = A$ and $C_B(\dot{X}^B|\dot{X}^S) = A$. Therefore $R_S(\dot{X}^S|\dot{X}^B) = \dot{X}^S \setminus A$, $R_B(\dot{X}^B|\dot{X}^S) = \dot{X}^B \setminus A$, so $X \setminus R_S(\dot{X}^S|\dot{X}^B) = \dot{X}^B$, $X \setminus R_B(\dot{X}^B|\dot{X}^S) = \dot{X}^S$. □

B.2 Proofs of Propositions 2 and 3

First, we prove that fixed points of an operator Φ (defined below) form a complete sublattice (Sublattice Theorem B.3) extending Theorems 7.3 and 7.5 from Fleiner (2003). Then we show two lattice properties for the terminal agents (Terminal Sublattice Theorem B.4 and Terminal Superiority Lemma B.6). We then use these three results to prove Propositions 2 and 3.

B.2.1 The sublattice property of fixed points

First note an immediate implication of the Laws of Aggregate Demand and Supply (LAD/LAS) that we have already noted in the proof of Lemma 1. If the choice functions of firm f satisfy LAD/LAS, for sets of contracts $Y' \subseteq Y \subseteq X_f^B$, and $Z \subseteq Z' \subseteq X_f^S$ (i.e. $(Y', Z') \sqsubseteq (Y, Z)$) then $|C_B^f(Y'|Z')| - |C_S^f(Z'|Y')| \leq |C_B^f(Y|Z)| - |C_S^f(Z|Y)|$.

For every firm f we define a weight function on the contracts in X_f, namely let $w(x) = 1$ if $x \in X_f^B$ and $w(x) = -1$ if $x \in X_f^S$. So $w(C^f(Y, Z)) = |C_B^f(Y|Z)| - |C_S^f(Z|Y)|$.[27] Therefore if C^f satisfies LAD/LAS, then $(Y', Z') \sqsubseteq (Y, Z)$ implies $w(C^f(Y', Z')) \leq w(C^f(Y, Z))$.

[27] The weight function can be defined more generally, see Fleiner (2003).

Let Y and Y' be subsets of X_f^B, Z and Z' are subsets of X_f^S. We denote the complement of Z in X_f^S with $\overline{Z} = X_f^S \setminus Z$. Define the operation $(Y,Z)\widetilde{\setminus}(Y',Z') = (Y \setminus Y', \overline{Z' \setminus Z})$. For a given firm f, we call a set function $R : 2^X_f \to 2^X_f$ a *w-contraction* if for every $(Y',Z') \sqsubseteq (Y,Z)$ pair, $w(R(Y,Z)\widetilde{\setminus}R(Y',Z')) \leq w((Y,Z)\widetilde{\setminus}(Y',Z'))$

Let us describe some properties of this $\widetilde{\setminus}$ operation:

Lemma B.1. For a firm f, let Y and Y' be subsets of X_f^B, Z and Z' are subsets of X_f^S such that $(Y',Z') \sqsubseteq (Y,Z)$. Then the following statements hold:

1. $w((Y,Z)\widetilde{\setminus}(Y',Z')) = w((Y,Z)) - w((Y',Z')) - |X_f^S|$.
2. For any (A,B) pair, $w((A,B)\widetilde{\setminus}(Y,Z)) \leq w((A,B)\widetilde{\setminus}(Y',Z'))$.
3. If $(Y,Z) \sqsubseteq (A,B)$ then the $w((A,B)\widetilde{\setminus}(Y,Z)) = w((A,B)\widetilde{\setminus}(Y',Z'))$ equality implies $(Y',Z') = (Y,Z)$.

Proof of Lemma B.1. Let us tackle each statement separately:

1. $w((Y,Z)\widetilde{\setminus}(Y',Z')) = |Y \setminus Y'| - |\overline{Z' \setminus Z}| = |Y| - |Y'| - |X_f^S| + |Z'| - |Z| = w((Y,Z)) - w((Y',Z')) - |X_f^S|$.
2. Since $Y \supseteq Y'$, this implies $A \setminus Y \subseteq A \setminus Y'$, and similarly $Z \subseteq Z'$ gives us $Z \setminus B \subseteq Z' \setminus B$, so $\overline{Z \setminus B} \supseteq \overline{Z' \setminus B}$, therefore $w((A,B)\widetilde{\setminus}(Y,Z)) = |A \setminus Y| - |\overline{Z \setminus B}| \leq |A \setminus Y'| - |\overline{Z' \setminus B}| = w((A,B)\widetilde{\setminus}(Y',Z'))$
3. If $w((A,B)\widetilde{\setminus}(Y,Z)) = w((A,B)\widetilde{\setminus}(Y',Z'))$ then equality must hold at $|A \setminus Y| = |A \setminus Y'|$ and $|\overline{Z \setminus B}| = |\overline{Z' \setminus B}|$. Since $Y' \subseteq Y \subseteq A$ and $Z' \supseteq Z \supseteq B$, we get that $Y = Y'$ and $Z = Z'$. □

Lemma B.2. Suppose that the choice function of $f \in F$ satisfies full substitutability and LAD/LAS. Then the rejection function R^f is $\sqsubseteq$-isotone and a w-contraction.

Proof of Lemma B.2. Let Y and Y' be subsets of X_f^B and Z and Z' are be of X_f^S, and moreover let $(Y',Z') \sqsubseteq (Y,Z)$.
We have seen earlier that R^f is $\sqsubseteq$-isotone, so $R^f(Y',Z') \sqsubseteq R^f(Y,Z)$. To prove that it is w-contraction, $w(R^f(Y,Z)\widetilde{\setminus}R^f(Y',Z')) + |X_f^S| = w(R^f(Y,Z)) - w(R^f(Y',Z')) = w((Y,Z) \setminus C^f(Y,Z)) - w((Y',Z') \setminus C^f(Y',Z')) = w(Y,Z) - w(C^f(Y,Z)) - w(Y',Z') + w(C^f(Y',Z')) \leq w(Y,Z) - w(Y',Z') = w((Y,Z)\widetilde{\setminus}(Y',Z')) + |X_f^S|$. We used that $w(C^f(Y',Z')) \leq w(C^f(Y,Z))$. If we subtract $|X_f^S|$ from both sides, we get that
$w(R^f(Y,Z)\widetilde{\setminus}R^f(Y',Z')) \leq w((Y,Z)\widetilde{\setminus}(Y',Z'))$, so R^f is indeed a w-contraction. □

We will work on the $(2^{(X,X)}, \widetilde{\cup}, \widetilde{\cap})$ lattice. We can imagine it as a network that contains exactly two (unrelated) copies of each contract (two half-contracts), one for the buyer and one for the seller of the contract.

Now the C^f choice functions of the firms are defined over disjoint set of contracts, so for every $Y \subseteq (X,X)$ we can define $C(Y) = \bigcup_{f \in F} C^f(Y_f)$. Similarly $R(Y) = \bigcup_{f \in F} R^f(Y_f)$. On this whole network, we call a set function $R : 2^{(X,X)} \to 2^{(X,X)}$ a *w-contraction* if for every firm f the corresponding R_f was a w-contraction.

Let us denote the set of the starting half-contracts (seller's side) with $X_F^S = \bigcup_{f\in F} X_f^S$, and the set of ending half-contracts (buyer's side) with $X_F^B = \bigcup_{f\in F} X_f^B$. Now $|X_F^S| = |X_F^B| = |X|$.

Let $Y \subseteq X_F^B$ and $Z \subseteq X_F^S$. The *dual* of (Y, Z) is what we get by switching the two parts. We denote it with $(Y, Z)^* = (Z, Y)$.

In this model let all the contracts in X_F^S have weight $w = -1$ and all contracts in X_F^B have weight $w = 1$.

Lemma B.3. If $F : 2^{(X,X)} \to 2^{(X,X)}$ is $\sqsubseteq$-isotone and a w-contraction then fixed points of F form a nonempty sublattice of $(2^{(X,X)}, \tilde{\cup}, \tilde{\cap})$.

Proof of Lemma B.3. By Theorem B.1, the set of fixed points is nonempty. Now let $U \subseteq (X, X)$ and $V \subseteq (X, X)$. Assume that $F(U) = U$ and $F(V) = V$. By monotonicity, $U \tilde{\cap} V = F(U) \tilde{\cap} F(V) \sqsupseteq F(U \tilde{\cap} V)$ and $U \tilde{\cup} V = F(U) \tilde{\cup} F(V) \sqsubseteq F(U \tilde{\cap} V)$. From the w-contraction property and Lemma B.1, we have that

$$w(U \tilde{\setminus} (U \tilde{\cap} V)) \geq w(F(U) \tilde{\setminus} F(U \tilde{\cap} V)) \geq w(U \tilde{\setminus} (U \tilde{\cap} V)),$$

$$w((U \tilde{\cup} V) \tilde{\setminus} U) \geq w(F(U \tilde{\cup} V) \tilde{\setminus} F(U)) \geq w((U \tilde{\cup} V) \tilde{\setminus} U),$$

hence an equality must hold throughout. Using the third part of Lemma B.1 we can see that $(U \tilde{\cap} V) = F(U \tilde{\cap} V)$ and $(U \tilde{\cup} V) = F(U \tilde{\cup} V)$ so they are also fixed points of F. ☐

Observation B.2. Consider two sets of contracts (Y, Z) and (Y', Z') , where $Y, Y' \subseteq X_F^B$ and $Z, Z' \subseteq X_F^S$ and $(X, X) \setminus (Y, Z) = (X \setminus Y, X \setminus Z)$. If $(Y', Z') \sqsubseteq (Y, Z)$, then $((X \setminus Y, X \setminus Z)^* \tilde{\setminus} (X \setminus Y', X \setminus Z')^*) = ((X \setminus Z) \setminus (X \setminus Z'), \overline{(X \setminus Y') \setminus (X \setminus Y)}) = ((Z' \setminus Z), \overline{(Y \setminus Y')}) = ((X, X) \setminus ((Y, Z) \tilde{\setminus} (Y', Z'))^*$.

Theorem B.3 (Sublattice Theorem)**.** Suppose that choice functions satisfy full substitutability and LAD/LAS. Then the fixed points of $\Phi(Y, Z) = (X \setminus R_S(Z|Y), X \setminus R_B(Y|Z))$ form a nonempty, complete sublattice of $(2^X \times 2^X, \tilde{\cup}, \tilde{\cap})$.

Proof of Theorem B.3. The $\Phi(Y, Z) = (X \setminus R_S(Z|Y), X \setminus R_B(Y|Z))$ function can be also written as $\Phi(Y) = ((X, X) \setminus R(Y, Z))^*$. Since R is $\sqsubseteq$-isotone, Φ is also $\sqsubseteq$-isotone. We need to show that Φ is a w-contraction. Suppose that $(Y', Z') \sqsubseteq (Y, Z)$. Using Observation B.2, $w(\Phi(Y, Z) \tilde{\setminus} \Phi(Y', Z')) = w(((X, X) \setminus R(Y, Z))^* \tilde{\setminus} ((X, X) \setminus R(Y', Z'))^*) = w(((X, X) \setminus (R(Y, Z) \tilde{\setminus} R(Y', Z')))^*) = w(R(Y, Z) \tilde{\setminus} R(Y', Z')) \leq w((Y, Z) \tilde{\setminus} (Y', Z'))$ because in Lemma B.2 we showed that R is a w-contraction.

Since Φ is $\sqsubseteq$-isotone and a w-contraction, Lemma B.3 gives that the fixed points of Φ form a sublattice of $(2^{(X,X)}, \tilde{\cup}, \tilde{\cap})$. ☐

B.2.2 Lattice for the terminal agents

The following path independence condition was introduced by Aizerman and Malishevski (1981). It has been deeply explored in many-to-one matching markets by Echenique and Yenmez (2015) and in many-to-many matching markets by Fleiner (2003) and Chambers and Yenmez (2017).

Lemma B.4. (Path Independence) If choice function $C^f : 2^X \to 2^X$ is same-side substitutable and satisfies IRC then $C^f(Y \cup Z) = C^f(Y \cup C^f(Z))$ holds for $Y, Z \subseteq X$.

Proof of Lemma B.4. Since C^f is same-side substitutable, $C^f(Y \cup Z) \subseteq (Y \cup C^f(Z))$. Using IRC we have that $C^f(Y \cup Z) \subseteq (Y \cup C^f(Z)) \subseteq (Y \cup Z)$ implies that $C^f(Y \cup Z) = C^f(Y \cup C^f(Z))$. □

Lemma B.5. Suppose that choice functions satisfy full substitutability and IRC. Then terminal superiority is a partial order on terminal-trail-stable outcomes.

Proof of Lemma B.5. We need to prove that $\preceq^S$ is reflexive, antisymmetric and transitive. Assume that A, A' and A'' are acceptable outcomes. As $C^f(A_f \cup A_f) = C^f(A_f) = A_f$ holds for each agent (and hence for each terminal seller) f, relation $\preceq^S$ is reflexive. If $A \preceq^S A' \preceq^S A$ then we have $A_f = C^f(A_f \cup A'_f) = A'_f$ holds for any terminal agent f, hence $A = A'$ and $\preceq^S$ is antisymmetric. For transitivity, assume that $A \succeq^S A' \succeq^S A''$. Using this and Lemma B.4, we get for any terminal agent f that

$$C^f(A_f \cup A''_f) = C^f(C^f(A_f \cup A'_f) \cup A''_f) = C^f(A_f \cup A'_f \cup A''_f) = C^f(A_f \cup C^f(A'_f \cup A''_f)) = C^f(A_f \cup A'_f) = A_f \,,$$

hence $A \succeq^S A''$ indeed holds. This completes the proof. □

Theorem B.4 (Terminal Sublattice Theorem)**.** If L is a nonempty complete sublattice of $(2^X \times 2^X, \widetilde{\cup}, \widetilde{\cap})$ then $L_{\mathcal{T}} = \{(Y_{\mathcal{T}}, Z_{\mathcal{T}}) : (Y, Z) \in L\}$ is a sublattice of $(2^{\mathcal{T}} \times 2^{\mathcal{T}}, \widetilde{\cup}, \widetilde{\cap})$.

Proof of Theorem B.4. For a given $(A_{\mathcal{T}}, B_{\mathcal{T}})$ there can be more than one inverse image in the original lattice. Let $(A^*, B^*) = \widetilde{\bigcup}\{(Y, Z) \in L : (Y_{\mathcal{T}}, Z_{\mathcal{T}}) \sqsubseteq (A_{\mathcal{T}}, B_{\mathcal{T}})\}$. Since L is a complete lattice with lattice operations $\widetilde{\cup}$ and $\widetilde{\cap}$, this means $(A^*, B^*) \in L$ and $(A^*_{\mathcal{T}}, B^*_{\mathcal{T}}) = (A_{\mathcal{T}}, B_{\mathcal{T}})$. We call it the *canonical inverse image* of $(A_{\mathcal{T}}, B_{\mathcal{T}})$, since this is the $\sqsubseteq$-greatest among all inverse images.

If $(A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(C_{\mathcal{T}}, D_{\mathcal{T}}) \in L_{\mathcal{T}}$, let us consider $(Y, Z) = (A^*, B^*) \widetilde{\cap} (C^*, D^*)$. Since $(Y, Z) \sqsubseteq (A^*, B^*)$ this implies $(Y_{\mathcal{T}}, Z_{\mathcal{T}}) \sqsubseteq (A^*_{\mathcal{T}}, B^*_{\mathcal{T}}) = (A_{\mathcal{T}}, B_{\mathcal{T}})$. Similarly $(Y_{\mathcal{T}}, Z_{\mathcal{T}}) \sqsubseteq (C_{\mathcal{T}}, D_{\mathcal{T}})$. We want to show that $(Y_{\mathcal{T}}, Z_{\mathcal{T}})$ is the greatest lower bound of $(A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(C_{\mathcal{T}}, D_{\mathcal{T}})$ in $L_{\mathcal{T}}$. We can see that $(Y^*, Z^*) \sqsubseteq (A^*, B^*)$ and $(Y^*, Z^*) \sqsubseteq (C^*, D^*)$ because (A^*, B^*) is defined by the union of a greater set. Therefore $(Y^*, Z^*) = (Y, Z)$.

Suppose there exists a $(E_{\mathcal{T}}, F_{\mathcal{T}}) \in L_{\mathcal{T}}$ such that $(E_{\mathcal{T}}, F_{\mathcal{T}}) \sqsubseteq (A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(E_{\mathcal{T}}, F_{\mathcal{T}}) \sqsubseteq (C_{\mathcal{T}}, D_{\mathcal{T}})$ but $(E_{\mathcal{T}}, F_{\mathcal{T}}) \not\sqsubseteq (Y_{\mathcal{T}}, Z_{\mathcal{T}})$. Then in the original lattice $(E^*, F^*) \sqsubseteq (A^*, B^*)$ and $(E^*, F^*) \sqsubseteq (C^*, D^*)$ but $(E^*, F^*) \not\sqsubseteq (Y^*, Z^*)$. But this is impossible since $(Y, Z) = (A^*, B^*) \widetilde{\cap} (C^*, D^*)$. So we have found a unique greatest common lower bound of $(A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(C_{\mathcal{T}}, D_{\mathcal{T}})$.

Similar argument can be made in order to find the lowest common upper bound of $(A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(C_{\mathcal{T}}, D_{\mathcal{T}})$. Let $(Y, Z) = (A^*, B^*) \widetilde{\cup} (C^*, D^*)$. Since $(Y, Z) \sqsupseteq (A^*, B^*)$ this implies $(Y_{\mathcal{T}}, Z_{\mathcal{T}}) \sqsupseteq (A^*_{\mathcal{T}}, B^*_{\mathcal{T}}) = (A_{\mathcal{T}}, B_{\mathcal{T}})$. Similarly $(Y_{\mathcal{T}}, Z_{\mathcal{T}}) \sqsupseteq (C_{\mathcal{T}}, D_{\mathcal{T}})$.

Suppose there exists a $(E_{\mathcal{T}}, F_{\mathcal{T}}) \in L_{\mathcal{T}}$ such that $(E_{\mathcal{T}}, F_{\mathcal{T}}) \sqsupseteq (A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(E_{\mathcal{T}}, F_{\mathcal{T}}) \sqsupseteq (C_{\mathcal{T}}, D_{\mathcal{T}})$ but $(E_{\mathcal{T}}, F_{\mathcal{T}}) \not\sqsupseteq (Y_{\mathcal{T}}, Z_{\mathcal{T}})$. Then in the original lattice $(E^*, F^*) \sqsupseteq (A^*, B^*)$ and $(E^*, F^*) \sqsupseteq (C^*, D^*)$ therefore $(E^*, F^*) \sqsupseteq (Y, Z)$, so $(E^*_{\mathcal{T}}, F^*_{\mathcal{T}}) = (E_{\mathcal{T}}, F_{\mathcal{T}}) \sqsupseteq (Y_{\mathcal{T}}, Z_{\mathcal{T}})$, which is a contradiction.

So we have found a unique lowest common upper bound of $(A_{\mathcal{T}}, B_{\mathcal{T}})$ and $(C_{\mathcal{T}}, D_{\mathcal{T}})$, so $(L_{\mathcal{T}}, \widetilde{\cup}, \widetilde{\cap})$ is indeed a lattice. □

Now we consider only the contracts sold by the terminal sellers. For any $Y \subseteq X$, let $Y_{\mathcal{S}} = \{x \in Y | s(x) \in \mathcal{T}\}$.

Given two trail-stable outcomes A and A', let us denote the canonical trail-stable pair (defined as at the end of Proof of Theorem 1) for A with $\dot{X}^B$ and $\dot{X}^S$, and the canonical trail-stable pair for A' with $\dot{X}'^B$ and $\dot{X}'^S$.

Lemma B.6 (Terminal Superiority Lemma)**.** Given two trail-stable outcomes A and A', $C^f(A_f \cup A'_f) = A_f$ for each teminal seller if and only if $\dot{X}^S_{\mathcal{S}} \supseteq \dot{X}'^S_{\mathcal{S}}$ and $\dot{X}^B_{\mathcal{S}} \subseteq \dot{X}'^B_{\mathcal{S}}$ holds. A similar statement holds for terminal buyers.

Proof of Lemma B.6. If f is a terminal seller, $C^f(\dot{X}^S) = A_f$ and $C^f(\dot{X}'^S) = A'_f$. Suppose that $\dot{X}^S_{\mathcal{S}} \supseteq \dot{X}'^S_{\mathcal{S}}$. By IRC, $A_f \subseteq A_f \cup A'_f \subseteq \dot{X}^S_f$ implies that $C^f(A_f \cup A'_f) = A_f$.

For the opposite direction, take a contract $x \in X_f$ such that $x \notin C^f(A'_f \cup x)$. We use Lemma B.5, $A \succeq^S A' \succeq^S x$, therefore $A \succeq^S x$, so $x \notin C^f(A_f \cup \{x\})$. When we define the stable pairs for A and A', if $x \in C^f(A'_f \cup \{x\})$ then $x \in \dot{X}^B$, if $x \notin C^f(A'_f \cup \{x\})$ then $x \in \dot{X}^S$. From the previous observation we can see that $\dot{X}^S_{\mathcal{S}} \supseteq \dot{X}'^S_{\mathcal{S}}$ and $\dot{X}^B_{\mathcal{S}} \subseteq \dot{X}'^B_{\mathcal{S}}$. The proof for terminal buyers is analogous. □

Proof of Proposition 2. In the proof of Theorem 1 we have seen that any fixed point $(\dot{X}^B, \dot{X}^S)$ of isotone mapping Φ on lattice L determines a trail-stable outcome A^X. Moreover, each trail-stable outcome A corresponds to at least one fixed point $(\dot{X}^B, \dot{X}^S)$ of Φ. From Theorem B.1, it follows that fixed points of Φ form a lattice, hence there is a $\sqsubseteq$-minimal fixed point $(\dot{Y}^B, \dot{Y}^S)$ and a $\sqsubseteq$-maximal one $(\dot{Z}^B, \dot{Z}^S)$. We show that trail-stable outcome A^Y is seller-optimal and A^Z is buyer-optimal. So assume that $A = A^X$ is a trail-stable outcome. As $(\dot{Y}^B, \dot{Y}^S) \sqsubseteq (\dot{X}^B, \dot{X}^S) \sqsubseteq (\dot{Z}^B, \dot{Z}^S)$, we have $\dot{Y}^B \subseteq \dot{X}^B \subseteq \dot{Z}^B$ and $\dot{Y}^S \supseteq \dot{X}^S \supseteq \dot{Z}^S$. Lemma B.6 implies that $C^f(A_f \cup A^Y_f) = A^Y_f$ and $C^f(A_f \cup A^Z_f) = A_f$ for any terminal seller f and $C_g(A_g \cup A^Y_g) = A_g$ and $C_g(A_g \cup A^Z_g) = A^Z_g$ for any terminal buyer g. So, by definition A is seller-superior to A^Y and A^Z is seller-superior to A. □

Proof of Proposition 3. In the proof of Theorem 1 we have seen that A is trail-stable if and only if there is canonical trail-stable pair $(\dot{X}^B, \dot{X}^S)$ such that $(\dot{X}^B, \dot{X}^S)$ is a fixed point of isotone mapping Φ and $A = \dot{X}^B \cap \dot{X}^S$. Moreover, if the choice functions satisfy LAD/LAS, then fixed points of Φ form a sublattice L of $(2^X \times 2^X, \widetilde{\cup}, \widetilde{\cap})$ by Theorem B.3. From Theorem B.4, the projection of the above lattice to the terminals, $L_{\mathcal{T}}$ is also a lattice under $\sqsubseteq$ and from Lemma B.6 this partial order coincides with $\preceq^S$. Therefore, the trail-stable outcomes form a lattice under terminal-superiority. □

B.3 Proofs of Propositions 4 and 5

Proof of Proposition 4. Follow the proof of Theorem 8 in Hatfield and Kominers (2012) word-for-word, only replacing "stable" with "trail-stable". □

Proof of Proposition 5. Follow the proof of Theorem 1 in Hatfield and Kojima (2009) (which was pointed out by Hatfield and Kominers (2012) for stable outcomes in supply chains). □

B.4 Proofs of Propositions 6 and 7

Our proof follows Ostrovsky (2008). First we investigate the restabilized outcome from A, which we play part in the proofs of both Propositions 6 and 7. Let A be an arbitrary trail-stable outcome in the original network, with a corresponding canonical trail-stable pair $(\dot{X}^B, \dot{X}^S)$. After the new terminal seller f' arrives, let X be the set of all contracts in the new network, and let us define $(X^{*B}, X^{*S}) = (\dot{X}^B, \dot{X}^S \cup X_{f'})$. In the following, we will use Φ according to the choice fuctions on the new network, so $(\dot{X}^B, \dot{X}^S)$ does not need to be a fixed point of Φ anymore.

Since $X_{f'} \cap X^{*B} = \emptyset$, for every firm $f \neq f'$, $R^f_S(X^{*S}|X^{*B}) = R^f_S(\dot{X}^S|\dot{X}^B)$ and $R^f_B(X^{*B}|X^{*S}) = R^f_B(\dot{X}^B|\dot{X}^S)$. For example, if f has a conctracts with f', contract $x = f'f$ was not offered for firm f in X^{*B} so it does not get rejected.

For firm f', $R^{f'}_S(X^{*S}|X^{*B}) = X_{f'} \setminus C^{f'}_S(X_{f'})$ and $R^{f'}_B(X^{*B}|X^{*S}) = \emptyset$.

Therefore $\Phi(X^{*B}, X^{*S}) = (\dot{X}^B \cup C^{f'}(X_{f'}), \dot{X}^S \cup X_{f'})$.

So $(X^{*B}, X^{*S}) \sqsubseteq \Phi(X^{*B}, X^{*S})$, and Φ is $\sqsubseteq$-isotone, so $\Phi(X^{*B}, X^{*S}) \sqsubseteq \Phi(\Phi(X^{*B}, X^{*S}))$ and so on. The lattice of all possible subset-pairs is finite, so there is a k such that $\Phi^k(X^{*B}, X^{*S}) = (\hat{X}^B, \hat{X}^S)$ is a fixed point. So $(X^{*B}, X^{*S}) \sqsubseteq \Phi(X^{*B}, X^{*S}) \sqsubseteq \Phi^k(X^{*B}, X^{*S}) = (\hat{X}^B, \hat{X}^S)$. Outcome $\hat{A} = \hat{X}^B \cap \hat{X}^S$ is trail-stable, and this is what we call the *restabilized outcome* from A.

Proof of Proposition 6. If f' is a terminal seller, and we start from outcome A_{max} and the $\sqsubseteq$-maximal pair $(\dot{Z}^B, \dot{Z}^S)$. Using the previous method, outcome $\hat{A} = \hat{Z}^B \cap \hat{Z}^S$ is the restabilized outcome from A. In the new network there exists a $\sqsubseteq$-maximal fixed point of Φ, namely (Z'^B, Z'^S), therefore $(\dot{Z}^B, \dot{Z}^S \cup X_{f'}) = (Z^{*B}, Z^{*S}) \sqsubseteq (\hat{Z}^B, \hat{Z}^S) \sqsubseteq (Z'^B, Z'^S)$. The trail-stable outcome corresponding to the maximal fixed point is $A'_{max} = Z'^B \cap Z'^S$. We have to show that A'_{max} is better for terminal buyers and worse for terminal sellers than the original A_{max}. If f is a terminal buyer, since (Z'^B, Z'^S) is fixed point of Φ and $(\dot{Z}^B, \dot{Z}^S)$ was fixed before the new agent arrived, $C^f(Z'^B) = A'_{f,max}$ and $C^f(Z^{*B}) = A_{f,max}$ and $Z^{*B} \subseteq Z'^B$ so from $C^f(Z'^B) \subseteq (A_{f,max} \cup A'_{f,max}) \subseteq Z'^B$ by IRC we obtain $C^f(A_{f,max} \cup A'_{f,max}) = A'_{f,max}$ so $A'_{f,max}$ is preferred by terminal buyers.

Similarly, if f is a terminal seller outside f', $C^f(Z'^S) = A'_{f,max}$ and $C^f(Z^{*S}) = A_{f,max}$ and $Z'^S \subseteq Z^{*S}$ so from $C^f(Z^{*S}) \subseteq (A_{f,max} \cup A'_{f,max}) \subseteq Z^{*S}$ by IRC we obtain $C^f(A_{f,max} \cup A'_{f,max}) = A_{f,max}$ so $A_{f,max}$ is preferred by terminal buyers. If f' is a terminal buyer then we can use the same proof with reversing the roles of buyers and sellers. □

Proof of Proposition 7. If f' is a terminal seller, and A is any trail-stable outcome in the original network, with canonical trail-stable pair $(\dot{X}^B, \dot{X}^S)$, then $(X^{*B}, X^{*S}) = (\dot{X}^B, \dot{X}^S \cup X_{f'}) \sqsubseteq (\hat{X}^B, \hat{X}^S)$. The restabilized outcome is $\hat{A} = \hat{X}^B \cap \hat{X}^S$, and similarly to the proof of Proposition 6 one can show that initial producers weakly prefer A to $\hat{A}$ and all end consumers (other than f') prefer $\hat{A}$ to A. If f' is a terminal buyer, preferences are the opposite. □

B.5 Proofs of Propositions 1, 9, 10, and 11

The following lemma shows that if a locally blocking trail intersects an agent several times, but he doesn't want to pick every contract in the locally blocking trail, then the agent will still select any of his upstream (downstream) contracts alongside some another one of his downstream (upstream) contracts in the locally blocking trail.

Lemma B.7. Suppose that choice functions satisfy full substitutability and IRC. Moreover, consider a set of contracts $Y \subset X$ and a set of contracts $\{x_1, x_2, \ldots, x_k, z_1, z_2, \ldots, z_k\}$ for agent f.

1. Assume that Y is acceptable and $x_1, x_2, \ldots, x_k \in X_f^B$ and $z_1, z_2, \ldots, z_k \in X_f^S$ such that $\{x_i, z_i\}$ is a (Y, f)-essential pair for any $1 \le i \le k$, but $\{x_1, x_2, \ldots, x_k, z_1, z_2, \ldots, z_k\}$ is not (Y, f)-acceptable. Then $\{x_i, z_j\}$ is a (Y, f)-essential pair for some $i \neq j$.

2. Now let's remove contract x_1. Assume that Y is acceptable and $x_2, \ldots, x_k \in X_f^B$ and $z_1, z_2, \ldots, z_k \in X_f^S$ such that z_1 is (Y, f)-acceptable, $\{x_i, z_i\}$ is a (Y, f)-essential pair for any $2 \le i \le k$, but $\{x_2, \ldots, x_k, z_1, z_2, \ldots, z_k\}$ is not (Y, f)-acceptable. Then $\{x_i, z_j\}$ is a (Y, f)-essential pair for some $i \neq j$.

3. Now let's remove contracts x_1 and z_k. Assume that Y is acceptable and $x_2, \ldots, x_k \in X_f^B$ and $z_1, z_2, \ldots, z_{k-1} \in X_f^S$ such that z_1 and x_k are (Y, f)-acceptable, $\{x_i, z_i\}$ is a (Y, f)-essential pair for any $2 \le i \le k-1$, but $\{x_2, \ldots, x_k, z_1, z_2, \ldots, z_{k-1}\}$ is not (Y, f)-acceptable. Then $\{x_i, z_j\} \neq \{x_k, z_1\}$ is a (Y, f)-essential pair for some $i \neq j$.

Proof of Lemma B.7. We proof each statement in turn.

1. Suppose, for example, that $z_j \notin C^f(Y \cup \{x_1, x_2, \ldots, x_k, z_1, z_2, \ldots, z_k\})$ for some j. Then from CSC, $z_j \notin C^f(Y \cup \{x_j, z_1, z_2, \ldots, z_k\})$. But $x_j \in C^f(Y \cup \{x_j, z_j\})$ because $\{x_j, z_j\}$ is (Y, f)-essential by assumption so from CSC we must have that $x_j \in C^f(Y \cup \{x_j, z_1, z_2, \ldots, z_k\})$. Since $\{x_j\}$ is not (Y, f)-acceptable, there must be a $z_l \in C^f(Y \cup \{x_j, z_1, z_2, \ldots, z_k\})$ so that $\{x_j, z_l\}$ is (Y, f)-acceptable for some $l \neq j$.

2. In the case that x_1 has been removed, suppose that $z_1 \notin C^f(Y \cup \{x_2, \ldots, x_k, z_1, z_2, \ldots, z_k\})$. Then, by CSC, we must have that $z_1 \notin C^f(Y \cup \{z_1, z_2, \ldots, z_k\})$ but this cannot hold if $\{z_1\}$ is (Y, f)-acceptable but none of the other $\{z_j\}$ contracts are (Y, f)-acceptable by IRC and SSS. Therefore, it must be that either x_j or z_j (for $2 \le j \le k$) is not chosen by $C^f(Y \cup \{x_2, \ldots, x_k, z_1, z_2, \ldots, z_k\})$. Following the argument in Part 1 above, there must be an (Y, f)-essential pair $\{x_i, z_j\}$.

3. Repeat the argument in Part 2. □

Before we prove Proposition 1, we will introduce a useful concept.

Definition B.1. A non-empty set of contracts Q is a *circuit* if its elements can be arranged in some order $(x_1, \ldots, x_M)$ such that $b(x_m) = s(x_{m+1})$ holds for all $m \in \{1, \ldots, M-1\}$ and $b(x_M) = s(x_1)$ where $M = |Q|$.

Proof of Proposition 1. To show that every stable outcome is trail-stable consider, towards a contradiction, a stable outcome A which is not trail-stable. Pick a locally blocking trail T. For every firm involved in T, if $T_f \nsubseteq C^f(A \cup T_f)$, then using Lemma B.7 there is a upstream-downstream pairs of contracts $x_j \in T$ and $z_l \in T$ such that $j \neq l$ and $\{x_j, z_l\}$ is (A, f)-acceptable. Select only these essential pairs and remove the other contracts. Note that the contracts that remain still form a locally blocking trail. Continue removing contracts by applying Lemma B.7 to the locally blocking trail until the remaining locally blocking trail forms circuit Z such that $Z_f \subseteq C^f(A \cup Z_f)$ for every firm f. This circuit is therefore a blocking set. Since $T \cap A = \emptyset$ and $Z \subseteq T$, $Z \cap A = \emptyset$. Therefore A is not stable, a contradiction. □

Lemma B.8. Suppose that choice functions satisfy full substitutability and IRC. If Y and Z are disjoint sets of contracts and f is an agent such that Z_f is (Y, f)-acceptable then for any contract z of Z_f^B one of the following options hold:

1. $\{z\}$ is (Y, f)-acceptable, or
2. there exists some $z' \in Z_f^S$ such that $\{z, z'\}$ is a (Y, f)-acceptable pair, or
3. there are $z_1, z_2, \dots, z_k \in Z_f^S$ such that both $\{z, z_1, z_2, \dots, z_k\}$ and $\{z_i\}$ (for $1 \le i \le k$) are (Y, f)-acceptable.

For $z \in Z_f^S$ an analogous statement holds.

Proof of Lemma B.8. We can suppose without loss of generality that $z \in X_f^B$.

1. From SSS, it follows that $z \in C^f(Y_f \cup Z_f^S \cup \{z\})$. Assume that $C^f(Y_f \cup Z_f^S \cup \{z\}) \cap Z_f^S = \emptyset$. Therefore, we have $C^f(Y_f \cup Z_f^S \cup \{z\}) \subseteq (Y_f \cup \{z\}) \subseteq (Y_f \cup Z_f^S \cup \{z\})$, so from IRC $z \in C^f(Y_f \cup \{z\})$, so $\{z\}$ is (Y, f)-acceptable.
2. If $\{z\}$ is not (Y, f)-acceptable then there are some contracts $\{z_1, z_2 \dots z_k\} = C^f(Y_f \cup Z_f^S \cup \{z\}) \cap Z_f^S$. If there exists an z_i such that is $\{z_i\}$ is not (Y, f)-acceptable, then using SSS again, we have $z_i \in C^f(Y_f \cup \{z, z_i\})$. Suppose $z \notin C^f(Y_f \cup \{z, z_i\})$, then $C^f(Y_f \cup \{z, z_i\}) \subseteq (Y_f \cup \{z_i\})$, and from IRC we have $C^f(Y_f \cup \{z, z_i\}) = C^f(Y_f \cup \{z_i\})$. But since $\{z_i\}$ is not (Y, f)-acceptable this is impossible, therefore $\{z, z_i\} \subseteq C^f(Y_f \cup \{z, z_i\})$, we achieved a (Y, f)-essential pair.
3. If all of $\{z_1, z_2 \dots z_k\}$ are (Y, f)-acceptable. □

A consequence of Lemma B.8 is that trail stability is stronger than weak trail stability under full substitutability.

Proof of Proposition 9. Consider a trail-stable outcome A. Suppose that A is not weakly trail-stable, i.e. there exists a sequentially blocking trail $T = \{x_1, x_2 \dots x_M\}$ for it. Without loss of generality, we may assume that (b)ii holds in Definition 8. The other case when (b)i holds can be proved analogously.

We are going to find indices $1 \le i_1 < i_2 < i_3 \dots i_l \le k$ such that

- $\{x_{i_1}\}$ is $(A, s(i_1))$-acceptable, and
- $b(x_{i_{m-1}}) = s(x_{i_m}) = f_m$ and $\{x_{i_{m-1}}, x_{i_m}\}$ is a (A, f_m)-essential pair for all $1 < m \le l$, and
- $\{x_{i_l}\}$ is $(A, b(i_l))$-acceptable.

So this subset of the trail forms a locally blocking trail T'.

In the sequentially blocking trail T, choose the last contract $x_i \in T$ such that $\{x_i\}$ is $(A, s(x_i))$-acceptable. There is at least one contract like this, since $\{x_1\}$ is $(A, s(x_1))$-acceptable by definition. Let $i_1 = i$.

Suppose we have already found $i_1 \dots i_m$ that satisfies our requirements. If $\{x_{i_m}\}$ is $(A, b(x_{i_m}))$-acceptable, we end the trail there, and let $l = m$. Otherwise, from the definition of sequentially blocking trails, for $f_{m+1} = b(x_{i_m})$, the ending subsequence $T_f^{\geq m} = \{x_m, ..., x_M\} \cap T_f$ is (A, f_{m+1})-acceptable. Using Lemma B.8, there is a contract $x_{i_{m+1}} \in T_f^{\geq m} \cap X_f^S$ such that $i_{m+1} > i_m$ and $\{x_{i_{m-1}}, x_{i_m}\}$ is a (A, f_m)-essential pair.

This way, we constructed a locally blocking trail, therefore A is not trail-stable. $\square$

C When do solution concepts coincide?

As we have seen, stable, trail-stable, and weakly trail-stable outcomes typically do not coincide in general trading networks. In this section, we introduce two sufficient conditions on agents' preferences that ensure that these solution concepts coincide.

C.1 Trail stability and weak trail stability

Our first restriction generalizes "flow-based" choice functions (Fleiner, 2014, Fleiner et al., 2018).

Definition C.1. Choice functions of $f \in F$ are *separable* if for any $A, W \subseteq X$ and $y \in X_f^B \setminus A$ and $z \in X_f^S \setminus A$, whenever A is (W, f)-acceptable, and $\{y, z\}$ is a (W, f)-essential pair, then $A \cup \{y, z\}$ is (W, f)-acceptable.

Separable choice functions impose a kind of independence on choices of pairs of upstream and downstream contracts. It says that whenever the firm chooses A alongside some set W and $\{y, z\}$ alongside W (but y and z would not be chosen separately alongside W since $\{y, z\}$ is a (W, f)-essential pair), then it would choose $A \cup \{y, z\}$ alongside W. Suppose signing A and $\{y, z\}$ are decisions made by separate units of the firm. Separable choice functions say that it can delegate the joint input-output decisions to the units because its overall choices do not require any coordination between the units. One natural example of separable choice functions is the following: suppose each firms totally orders individual upstream contracts and individual downstream contracts. Whenever a firm is offered k downstream and l upstream contracts, it chooses the z best upstream and the z best downstream contracts where $z = \min(k, l)$.

We now pin down the role of separability for weakly trail-stable and trail-stable outcomes.

Proposition 10. Suppose that choice functions satisfy full substitutability, separability, and IRC. Then an outcome is trail-stable if and only if it is weakly trail-stable.

Proof of Proposition 10. Proposition 9 implies that if outcome A is trail-stable then A is also weakly trail-stable. So assume that outcome A is weakly trail-stable. If A is not trail-stable then there is a locally blocking trail T to A. The separability property of the choice functions imply that T is a sequentially blocking trail, contradicting the weak trail stability of A. So A is trail-stable. □

Separability is crucial for the correspondence between trail-stable and weakly trail-stable outcomes. Separability ensures that all sequentially blocking trails are locally blocking trails. Under separability all properties of trail-stable outcomes apply to weakly trail-stable outcomes.[28] Note that in Example 4, j's preferences were not separable and weak trail-stable outcome did not coincide with trail-stable outcomes.

[28] We also conjecture that in any trading network X if choice functions of F satisfy full substitutability and *only* LAD/LAS then the terminal-*weakly*-trail-stable contract sets form a lattice under terminal superiority, but leave this for future work.

C.2 Trail stability and stability

Separability is not enough to ensure that trail stable outcomes coincide with stable outcomes, since stable outcomes might not exist under full substitutability and separability (see Example 1). We turn to another preference restriction.

Definition C.2. Choice functions of $f \in F$ are *simple* if there exists an "intensity" mapping $w : X_f \to \mathbb{R}$ such that whenever A is a (W, f)-acceptable set for some acceptable set W of contracts, then for every $y \in X_f^B \cap A$ there exists $z \in X_f^S \cap A$ such that $w(y) > w(z)$ holds.

One example of choice functions which are simple are the following: if the agent is offered a set of contracts, he picks the upstream contract y with the highest intensity and a downstream contract z with the lowest intensity (as long as the intensity of the y is greater than of z, otherwise he picks nothing). For example, if the intensity mapping w represents the per-unit price of the contract, then the condition says that the firm only signs a pair of contracts if the price in the downstream contract is greater than the price in the upstream contract, while picking the highest-price downstream contract and the lowest-price upstream contract.

Proposition 11. Suppose that choice functions satisfy full substitutability, simplicity, and IRC. Then an outcome is stable if and only if it is weakly trail-stable.

Under simplicity all properties of trail-stable outcomes (including existence!) apply to stable outcomes.

Proof of Proposition 11. Proposition 9 implies that if outcome A is stable then A is also trail-stable. Assume that outcome A is trail-stable, but not stable, so it has a blocking set Z.

Case 1: Suppose that for any $z \in Z$ contract $\{z\}$ is neither $(A, s(z))$-acceptable nor $(A, b(z))$-acceptable. Then using Lemma B.8 we can find a circuit $Q = \{z_1, z_2, \ldots z_k\} \subseteq Z$ such that $\{z_i, z_{i+1}\}$ is an $(A, b(z_i))$-essential pair for every $1 \leq i \leq k$ and $\{z_k, z_1\}$ is an $(A, b(z_k))$-essential pair. Since every $\{z_i, z_{i+1}\}$ an $(A, b(z_i))$-acceptable set by itself, as choice functions are simple, intensity function w must strictly decrease along circuit Q, which is impossible.

Case 2: Suppose that for every $z \in Z$, $\{z\} \subseteq Z$ is A-acceptable. Suppose that $\{z_1\}$ is $(A, s(z_1))$-acceptable. From Lemma B.8 we can find a trail $\{z_2, z_3 \ldots z_k\} \subseteq Z$ such that for every z_i, either $\{z_i, z_{i+1}\}$ is a $(A, b(z_i))$-essential pair, (therefore $w(z_i) > w(z_{i+1})$) or there are some $y_1 \ldots y_l$ such that $b(y_j) = s(z_i)$ for all $1 \leq j \leq l$ and $\{z_i, y_1 \ldots y_l\}$ is $(A, b(z_i))$-acceptable. From the simplicity property there is a y_j such that $w(z_i) > w(y_j)$, this y_j contract will be z_{i+1}. The trail terminates at the first occasion when $\{z_i\}$ is $(A, b(z_i))$-acceptable.

Since the intensity strictly decreases, we cannot get back to a contract used earlier in the trail, so the trail must terminate. Let us pick a contract $\{z_i\}$ in the trail such that it is the last one which is $(A, s(z_i))$-acceptable, and then choose the smallest j such that $j \geq i$ and $\{z_j\}$ is $(A, b(z_j))$-acceptable. From Lemma B.8, the trail from z_i to z_j is locally blocking, so outcome A is not trail-stable. □

D Path stability

We can also weaken trail stability by insisting that trails only include any agent at most once so firms only have one opportunity to recontract during a deviation. A trail T is a *path* if all the agents $F(T)$ involved in the trail are distinct.

Definition D.1. An outcome $A \subseteq X$ is *path-stable* if

1. A is acceptable.
2. There is no path $P = \{x_1, x_2, \ldots, x_M\}$, such that $P \cap A = \emptyset$ and
   (a) $\{x_1\}$ is (A, f_1)-acceptable for $f_1 = s(x_1)$, and
   (b) $\{x_{m-1}, x_m\}$ is (A, f_m)-acceptable for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq M$ and
   (c) $\{x_M\}$ is (A, f_{M+1})-acceptable for $f_{M+1} = b(x_M)$.

Path-stable outcomes rule out consecutive pairwise deviations along paths. Since every path is a trail, every trail-stable outcome is path-stable. In acyclic networks every trail is also path, so path-stable, weakly trail-stable and trail-stable outcomes coincide with stable outcomes (Hatfield and Kominers, 2012). However, as the example below shows, path stability is weaker than weak trail stability (and hence weaker than trail stability) in general trading networks under full substitutability. This is intuitive because paths allows the firms to appear in the blocking set only once therefore they rule out fewer possible blocks.

Example 5 (Path-stable outcomes are not necessarily trail-stable)**.** Consider agents and contracts described in Examples 1 and 2, and Figure 1. Agents have the following fully substitutable preferences:

$$\begin{aligned}
\succ_m &: \{w\} \succ_m \emptyset \\
\succ_i &: \{x\} \succ_i \emptyset \\
\succ_k &: \{z, y\} \succ_k \emptyset \\
\succ_j &: \{w, x, z, y\} \succ_j \{w, z\} \succ_j \{y, x\} \succ_j \{y, z\} \succ_j \emptyset
\end{aligned}$$

The empty set is preferred to any other set of contracts.
Now, for outcome $\emptyset$, the trail $\{w, z, y, x\}$ is locally blocking, but there is no blocking path for $A = \emptyset$. Outcome $\{z, y\}$ is, however, blocked by path $\{w, x\}$. Therefore the trail-stable outcome is $\{w, z, y, x\}$ and the path-stable outcomes are $\emptyset$ and $\{w, z, y, x\}$.

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