

Competitive Rumour Spread in Social Networks

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Abstract

We consider a setting in which two firms compete to spread rumors in a social network. Firms seed their rumors simultaneously and rumors propagate according to the linear threshold model. Using the concept of *cascade centrality* introduced by [19], we provide a sharp characterization of networks that admit pure-strategy Nash equilibria (PSNE). We provide tight bounds for the efficiency of these equilibria and for the inequality in firms' equilibrium payoffs. The model can be extended to rumors of different persuasiveness.

Keywords: cascades, diffusion, social networks, competition, rumors.¹

1 Introduction

A rumor, according to the *Oxford English Dictionary*, is “an unverified or unconfirmed statement or report circulating in a community”. These rumors can damage company reputations, change voting behavior or spread hype about a product before its launch. Many firms now have detailed information about their customers' social networks and use this information to spread rumors. Therefore, it is not surprising that firms often try to spread competing rumors. Once a consumer spreads a rumor, they cannot usually take the rumor back, so the rumor is, at least in the short run, irreversible. This creates an incentive for firms to hunt for “early influencers” in order to maximize the spread of rumors in the social network.

In our model, there are two firms and two competing rumors. Each firm has a (unit) budget, which it can spend to “seed” the network with initial spreaders. Firms seed their rumors simultaneously and then the rumors diffuse according to the linear threshold model [15]. Each consumer has a spreading threshold drawn from the uniform distribution. A consumer decides

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to spread one of the rumors as soon as the proportion of his friends who have spread either rumor is greater than the threshold. Since the persuasiveness of the rumors is the same and consumers can spread only one rumor at a time, we assume that (once a spreading decision has been made) consumers are more likely to spread the rumor that more of their most recent gossiping friends have spread. A firm’s payoff is the expected number of consumers who have spread its rumor.

Even when each firm has a unit budget i.e. can only seed to exactly one initial spreader, there may be no pure-strategy Nash equilibrium (PSNE) in general social networks. Informally, a PSNE may fail to exist if the social network is very symmetric and has many cycles. However, using *cascade centrality*, we are able to characterize PSNEs whenever they exist. Cascade centrality of an agent in a network measures the expected number of consumers who spread that rumor when the agent alone is the seed and there are no other competing rumors [19]. In sufficiently asymmetric networks, there are certain agents – those with the highest cascade centrality – who are obvious candidates for seeding in a pure-strategy Nash equilibrium. Using the count of paths between agents, we give an explicit characterization of PSNEs when each firm has a unit budget. In general, PSNEs are inefficient: a social planner, who wants to maximize the total number of spreaders of either rumor can outperform competing firms. The inefficiency is entirely due to the path dependence of the diffusion process and comes from the externality imposed on the competing firm by the other firm’s seed. However, we show that the price of anarchy [18] – the ratio of the socially optimal number of spreaders to the number of spreaders in the worst PSNE – is at most 1.5.² The highest ratio of equilibrium payoffs, known as the budget multiplier [12], is at most 2. Moreover, we illustrate that both of these bounds are tight.

To illustrate the tractability of our framework, we also consider the diffusion game in tree networks (i.e. networks without cycles). The cascade centrality of an agent in a tree simply equals his degree plus one [19]. We show that PSNEs always exist in trees, and, for the case of unit budgets we provide a complete characterization of their structure and efficiency properties. Our characterization allows us to compute all PSNEs and the optimal seeds in $O(n^3)$ in the worst case by the Floyd-Warshall shortest-path algorithm [10].

Following the seminal work on influence maximization in networks by [15], there have been a number of papers on the algorithmic aspects of competitive rumor cascades [17, 14, 13, 11]. Our model, however, differs from models of product diffusion with network externalities [3, 4, 5, 6, 7, 8, 9, 12, 20, 22, 24]. Moreover, in a many competitive product diffusion models (e.g. [12]), agents consider *all* their neighbors when make both the product adoption decision and the product selection decision. In our model, only the decision to spread the rumor depends on the fraction of his neighbors who are spreading the rumor. But the decision of which of the two rumors to spread depends only on the *latest* neighbors who are spreading the rumor.³ This modeling assumption captures the observation that people are likely to use the latest information (e.g. the top of their Twitter feeds) to make a decision about which rumor to spread (perhaps due to limited attention as in [25]). Many of these models, unlike ours, involve

²[13] show that for a more general case of our model the lower bound on the price of anarchy is $\frac{4}{3}$ and the tight upper bound is 2.

³[11] call this a *linear switching function*.

prices (e.g. [7]), multiple firms (e.g. [13]) or directed, weighted graphs (e.g. [11]). Our aim is to use the simplest possible framework to shed an analytical light on the structure of equilibria in competitive diffusion games and illustrate the usefulness of cascade centrality.

Our model is also different from opinion dynamics models, such as [1], in which opinions may be updated over time. Our papers may be viewed as complementary: the present model is better suited to short-run rumor spread, whereas the model in [1] describes long-run opinion formation. Other approaches for modeling rumor spread use methods from statistical physics [21], epidemiology [23] and agent-based models [25].

This paper is organized as follows. In the next section, we describe the model, the spread dynamics and a host of analytical tools developed in part by [19]. In Section 3, we give an example of a network without a PSNE and then fully characterize general networks in which PSNEs exist. We give tight bounds on the budget multiplier and the price of anarchy. In Section 4, we focus on the duopoly in trees and prove that PSNEs always exist. In Section 5, we describe an extension of the model. We conclude in Section 6. All proofs are in the Appendix.

2 Model

2.1 Preliminaries

Let $G(V, E)$ be a simple (unweighted and undirected), connected graph with a set of n agents $V := \{1, \dots, n\}$ and a set of m links E .⁴ We denote the neighbors of $i \in V$ as $N_i(G) := \{j | (j, i) \in E\}$ and the degree of i as $d_i := |N_i(G)|$. A *threshold* for agent i is a random variable Θ_i drawn independently from a probability distribution with support $[0, 1]$. The associated multivariate probability density function for all the nodes in the graph is $f(\boldsymbol{\theta})$. Each agent $i \in V$ is assigned a threshold θ_i . Let's define the threshold profile of agents as $\boldsymbol{\theta} := (\theta_i)_{i \in V}$. A network $G_{\boldsymbol{\theta}}$ is a graph endowed with a threshold profile.

There are two firms, A and B , which spread two different, competing rumors a and b respectively. Agents on the social network can either spread rumor a or rumor b or remain non-spreaders. The state of agent i at time t is denoted $x_i(t) \in \{0, \alpha, \beta\}$. Denote by $S_t^A(G_{\boldsymbol{\theta}})$ and $S_t^B(G_{\boldsymbol{\theta}})$ the sets of additional spreaders of rumors a and b (i.e. agents in state α or β) in network $G_{\boldsymbol{\theta}}$ at time t respectively.

2.2 Dynamics

At time $t = 0$, all agents start out in state 0 and firms simultaneously choose a subset of agents $S_0^A, S_0^B \subseteq V$ as seed sets for their rumor. Some agents may be selected by both firms so we resolve any overlap in the seed sets randomly:

$$i \in S_0^A \cap S_0^B \Rightarrow \begin{cases} i \in S_0^A(G_{\boldsymbol{\theta}}) & \text{w.p. } \frac{1}{2} \\ i \in S_0^B(G_{\boldsymbol{\theta}}) & \text{w.p. } \frac{1}{2} \end{cases}$$

⁴An extension of the model to a directed graph is fairly straightforward and does not substantially affect the analysis.

We can therefore consider the dynamic process starting from *disjoint* seed sets $S_0^A(G_\theta)$ and $S_0^B(G_\theta)$.

In time period $t = 1$, any non-seed agent $i \in V \setminus [S_0^A(G_\theta) \cup S_0^B(G_\theta)]$ will decide to spread one of the rumors if the proportion of spreaders of either rumor among his neighbors exceeds his threshold:

$$\frac{|[S_0^A(G_\theta) \cup S_0^B(G_\theta)] \cap N_i(G_\theta)|}{|N_i(G_\theta)|} \geq \theta_i \Rightarrow \begin{cases} i \in S_1^A(G_\theta) & \text{w.p. } \frac{|S_0^A(G_\theta) \cap N_i(G_\theta)|}{|[S_0^A(G_\theta) \cup S_0^B(G_\theta)] \cap N_i(G_\theta)|} \\ i \in S_1^B(G_\theta) & \text{w.p. } \frac{|S_0^B(G_\theta) \cap N_i(G_\theta)|}{|[S_0^A(G_\theta) \cup S_0^B(G_\theta)] \cap N_i(G_\theta)|} \end{cases}$$

Conditional on exceeding the spread threshold, the probability of spreading a particular rumor is equal to the proportion of neighbors who have spread that rumor. In general, for a given period $t \geq 0$ and any non-spreader agent $i \in V \setminus [\{\cup_{\tau=0}^{t-1} S_\tau^A(G_\theta)\} \cup \{\cup_{\tau=0}^{t-1} S_\tau^B(G_\theta)\}]$ will spread a rumor according to the following rule

$$\frac{|[\{\cup_{\tau=0}^{t-1} S_\tau^A(G_\theta)\} \cup \{\cup_{\tau=0}^{t-1} S_\tau^B(G_\theta)\}] \cap N_i(G_\theta)|}{|N_i(G_\theta)|} \geq \theta_i \Rightarrow \begin{cases} i \in S_t^A(G_\theta) & \text{w.p. } \frac{|S_{t-1}^A(G_\theta) \cap N_i(G_\theta)|}{|[\{S_{t-1}^A(G_\theta)\} \cup \{S_{t-1}^B(G_\theta)\}] \cap N_i(G_\theta)|} \\ i \in S_t^B(G_\theta) & \text{w.p. } \frac{|S_{t-1}^B(G_\theta) \cap N_i(G_\theta)|}{|[\{S_{t-1}^A(G_\theta)\} \cup \{S_{t-1}^B(G_\theta)\}] \cap N_i(G_\theta)|} \end{cases}$$

Note that, while the decision to spread a rumor is made using the *all* neighbor-spreaders, the selection of the rumor is made only using the *latest* neighbor-spreaders. For a given network G_θ , define the fixed point of the process as a union of disjoint sets of agents in states α and β : $(S_0^A \cup S_0^B) = S^A(G_\theta, S_0^A, S_0^B) \cup S^B(G_\theta, S_0^A, S_0^B) \Rightarrow S_t^A(G_\theta) = \emptyset \wedge S_t^B(G_\theta) = \emptyset$ for all $t > 0$. This process converges almost surely to a random set, extending the argument in [2].

We are interested in the probability that an agent spreads rumor a in a given graph G with firms' seeds S_0^A and S_0^B when thresholds are drawn from distribution $f(\theta)$. To calculate this probability, let's take the expectation with respect to the threshold distribution.

$$\mathbb{P}_i^A(G, S_0^A, S_0^B) = \int_{\mathbb{R}^n} |S^A(G_\theta, S_0^A, S_0^B) \cap \{i\}| f(\theta) d\theta$$

This means that the expected number of spreaders of rumor A in a given graph G with firms' seeds S_0^A and S_0^B is simply:

$$\mathbb{E}[S^A(G, S_0^A, S_0^B)] = \int_{\mathbb{R}^n} |S^A(G_\theta, S_0^A, S_0^B)| f(\theta) d\theta = \sum_{i=1}^n \mathbb{P}_i^A(G, S_0^A, S_0^B)$$

The expressions for rumor b are analogous. Our setup ensures that

$$\mathbb{P}_i^A(G, S_0^A, S_0^B) + \mathbb{P}_i^B(G, S_0^A, S_0^B) \leq 1.$$

In order to focus our discussion on the role of the network structure in competitive rumor spread, we do not consider general threshold distributions. Instead we focus on the case where firms have a Laplacian prior about the agents' thresholds. [16].

Assumption 1. For any G_θ and every $i \in V$, $\Theta_i \sim \mathcal{U}(0, 1)$ and independent.

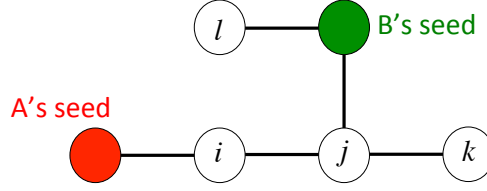


Figure 1: Unit budget duopoly in a social network

This assumption was first introduced and used by [15] to study influence maximization. [19] showed that it offers a great deal of tractability in the analysis of cascade processes. In fact, [16] convincingly argues that this assumption on the distribution thresholds is not restrictive. We therefore drop θ subscript and henceforth $G \equiv G_\theta$.

Figure 1 provides an example a simple network, which illustrates what outcomes the dynamics of the diffusion process can produce for the firms. Let's consider the possible outcomes from the perspective of firm A . Rumor a reaches i with probability $\frac{1}{2}$.⁵ If j doesn't spread b then its threshold must be above $\frac{1}{3}$. But for a to spread to j its threshold must be between $\frac{1}{3}$ and $\frac{2}{3}$. If that is the case (w.p. $\frac{1}{3}$), then conditional on reaching i , a reaches j and k with probability 1. Hence, A 's payoff is $\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{5}{6}$. A can never reach l because of B 's seed. Rumor b will certainly spread to l . The probability of spreading to j is $\frac{1}{3}$. Once b spread to j , it will certainly spread to k , moreover it will certainly spread to i if a does not spread to i (w.p. $\frac{1}{2}$). So, B 's payoff is $1 + \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 1 = 1\frac{5}{6}$.

3 Analytical tools

In order to capture path dependence of the rumor spread, we first introduce paths.⁶

Definition 1. A sequence of nodes $P = (i_0, \dots, i_k)$ on a graph G is a path if $i_j \in N_{i_{j-1}}(G)$ for all $1 \leq j \leq k$ and each $i_j \in P$ is distinct.

Since every node in the path is distinct, so is every edge. In general, there may be multiple paths between any two nodes. Figure 2 shows one possible path between nodes i and j . There are three paths between i and j in total in this network.

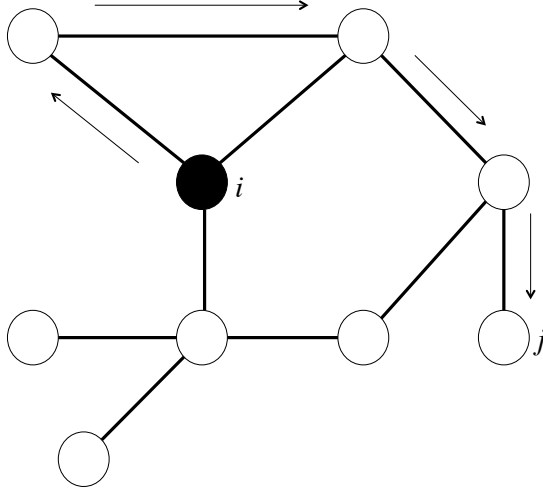
Definition 2. For a path P , the degree sequence along P is denoted $(d_i(G))_{i \in P}$. The degree sequence product along P is:

$$\chi_P := \prod_{i \in P} \frac{d_i(G)}{d_{i_0}}$$

⁵Note that this is the case because of our assumption that only the latest neighbor-spreaders matter for the choice of the rumor. If both all existing neighbors-spreader were used after passing the threshold – as in many product diffusion models – the probability of a reaching i would be higher.

⁶Some graph theory textbooks refer to “paths” as “simple paths”. Since we only use “simple paths” in this paper, we refer to them as “paths” without any ambiguity.

Figure 2: A path between two nodes



For any G and i , let $\mathcal{P}(i, j)$ be the set of all paths beginning at i and ending at $j \in V \setminus \{i\}$. Cascade centrality of a node i , formalized in [19], tells us the expected number of spreaders of its rumor when either firm seeds to i as a monopolist i.e. in absence of a competing firm.

Definition 3. *Cascade centrality of node i in graph G is the expected number of spreader in that graph given i is the seed, namely*

$$\mathcal{C}_i(G) := \mathbb{E}[S(G, \{i\})] = 1 + \sum_{j \in V} \sum_{P \in \mathcal{P}_{ij}} \frac{1}{\chi_P}$$

Cascade centrality captures the importance of a node in the network by measuring how large a cascade it induces when it alone is the seed. In order to see how cascade centrality can be used to generate insights into cascade processes, we prove a useful decomposition result. We first introduce a loop, which is a sequence of nodes (i.e. a walk) that bends back on itself exactly once.

Since the reference to G is usually unambiguous, we refer to cascade centrality of agent $\mathcal{C}_i(G)$ as \mathcal{C}_i and to degree of an agent $d_i(G)$ as d_i . For $i \neq j$, let us denote $\Xi(i, j)$ as the set of all paths that begin at i and include (but do not necessarily end at) j . Now, we define a new, but related, notion that will play an important role in this paper.

$$\epsilon(i, j) = \sum_{P \in \Xi(i, j)} \frac{1}{\chi_P}$$

$\epsilon(i, j)$ describe the extent to which node j interferes with the cascade emanating from node i . Note that $\epsilon(i, j) \neq \epsilon(j, i)$ unless $d_j = d_i$, so a node with a higher degree interferes more with a node of a lower degree than vice versa.

4 Results

4.1 Game and equilibrium

We focus on a duopoly with unit budgets.

Assumption 2. *Each firm has one initial seed i.e. $|S_0^A| = |S_0^B| = 1$.*

The action space of players A and B is hence denoted $\Sigma := \Sigma_A \times \Sigma_B := V \times V$; action profile $\sigma := (\sigma_A, \sigma_B)$. Action profiles are therefore simply (ordered) pairs of nodes (agents) in the network.

A firm's payoff is simply the expected number of spreaders of its rumor given the action of the other firm.⁷ For a payoff profile $\pi := (\pi_A(\sigma), \pi_B(\sigma))$, we can define the game $\Gamma := (\Sigma, \pi)$.

Define $\mathcal{P}^*(i, j) \subseteq \mathcal{P}(i, j)$ as the set of all paths from i to j that pass through the set of seed nodes $S_0^A \cup S_0^B$ at most once. Using our results from [19], Proposition 1 shows that we can express firm A 's payoff in two complementary ways.

Proposition 1. *Consider a duopoly with unit budgets Γ . The expected number of spreaders of rumor a (i.e. firm A 's payoff) is*

$$\pi_A(\sigma_A, \sigma_B) := \sum_{j \in V} \sum_{P \in \mathcal{P}^*(\sigma_A, j)} \frac{\iota}{\chi_P}$$

where

$$\iota = \begin{cases} 1/2 & \text{if } \sigma_A = \sigma_B \\ 1 & \text{otherwise} \end{cases}$$

Alternatively, we can express firm A 's payoff as

$$\pi_A(\sigma_A, \sigma_B) = \begin{cases} \mathcal{C}_{\sigma_A}/2 & \text{if } \sigma_A = \sigma_B \\ \mathcal{C}_{\sigma_A} - \epsilon(\sigma_A, \sigma_B) & \text{otherwise} \end{cases}$$

The expressions are analogous for $\pi_B(\sigma_A, \sigma_B)$, the payoff of firm B . In Figure 1, the reader can verify that $\mathcal{C}_{\sigma_A} = 2$, $\epsilon(\sigma_A, \sigma_B) = \epsilon(\sigma_B, \sigma_A) = \frac{1}{6}$. Figure 3 illustrates how to calculate payoffs for firms in a more complex social network. Firm A clearly has a higher payoff. This is mainly because it blocks off firm B 's access to two nodes in bottom-left part of the network and gets them to spread rumor a with probability 1. Except for these nodes and for the node in the bottom-right corner, either firm can reach all other nodes.

We want to focus on the pure-strategy Nash equilibria of this game.⁸

Definition 4. *A profile of actions $\sigma^* := (\sigma_A^*, \sigma_B^*) \in \Sigma$ is a pure-strategy Nash equilibrium of Γ if:*

- $\pi_A(\sigma_A^*, \sigma_B^*) \geq \pi_A(\sigma_A, \sigma_B^*)$ for all actions $\sigma_A \in \Sigma_A$

⁷An interesting extension is to consider a payoff function that makes a firm better off when the competitor's rumor spreads less.

⁸Since the number of firms and the strategy spaces are finite, a mixed-strategy Nash equilibrium always exists.

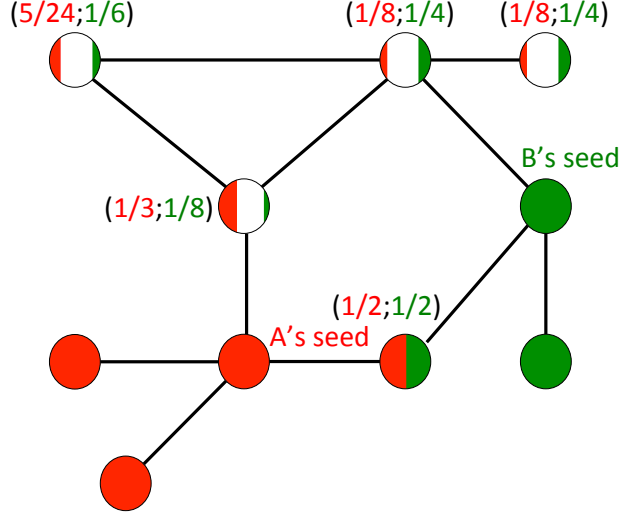


Figure 3: Unit budget duopoly in a social network

$$- \pi_B(\sigma_A^*, \sigma_B^*) \geq \pi_B(\sigma_A^*, \sigma_B) \text{ for all actions } \sigma_B \in \Sigma_B$$

Define Σ^* as the set of all pure-strategy Nash equilibria of the game.

4.2 Competition on general networks

In general, social networks do not admit a PSNE. Figure 4 gives an example of such a network. Note that this network is very symmetric and contain many cycles so the payoffs from seeding in its central nodes i , j , and k are very similar. The agent with the highest cascade centrality is i . The pure-strategy best response to a firm's seeding i is to seed k . But the best response to k is j and, in turn, the best response to j is to seed i . Therefore, the only equilibria in this social network are mixed-strategy. Figure 4 does not illustrate a pathological example; non-existence of PSNE in our model as in many other competitive diffusion models is generic [7].⁹

We now state the main result of this paper, which establishes necessary and sufficient conditions on the network for the existence of a PSNE. We characterize both types of PSNE: in which firms seed the same agent and in which firms seed different agents.

Theorem 1. *Consider a duopoly with unit budgets Γ . Then Γ admits at least one PSNE if and only if at least one of the following conditions on G is satisfied:*

1. *There exists $i \in V$ such that, for any $j \in V \setminus \{i\}$:*

$$\bullet \frac{C_i}{C_j} \geq 2 - 2 \cdot \left(\frac{\epsilon(j, i)}{C_j} \right)$$

- (a) *There exist $i, j \in V$ such that, $C_i \geq C_j$ and for any $k \in V \setminus \{i, j\}$:*

⁹We provide the full calculation of the payoffs in the Appendix.

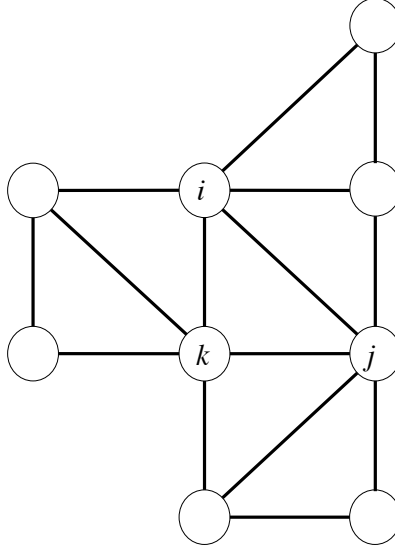


Figure 4: Example of a network that does not admit a PSNE

- $\frac{C_i}{C_k} \geq 1 + \frac{\epsilon(i, j) - \epsilon(k, j)}{C_k}$
- $\frac{C_j}{C_k} \geq 1 + \frac{\epsilon(j, i) - \epsilon(k, i)}{C_k}$
- $\frac{1}{2} + \frac{\epsilon(i, j)}{C_j} \leq \frac{C_i}{C_j} \leq 2 - 2 \cdot \left(\frac{\epsilon(j, i)}{C_j} \right)$

If Condition 1 is satisfied then there exists a $\sigma^* = (i, i)$ PSNE and if Condition 2 is satisfied, then there exists a $\sigma^* = (i, j)$ (and $\sigma^* = (j, i)$ by symmetry) PSNE.

Although the conditions set out in Theorem 1 appear involved, they are rather intuitive.¹⁰ Let us first focus on Condition 1. When firms seed the same agent i , their payoff is 0 with probability $\frac{1}{2}$ and C_i with probability $\frac{1}{2}$. To ensure that it is worthwhile for both firms to take that gamble, the cascade centrality of the node has to be sufficiently high. Indeed, it is sufficient that the cascade centrality of the agent with the second highest cascade centrality is at most half of C_i as the following corollary summarizes.

Corollary 1. $\sigma^* = (i, i)$ is a PSNE of Γ if $\frac{C_i}{C_j} \geq 2$ for any $j \neq i$.

Condition 2 is necessary and sufficient for the existence of a $\sigma^* = (i, j)$ PSNE with two different seeds. The first two parts of the condition say that the cascade centralities of i and j need to be sufficiently high compared to other agents in the network. The final condition ensures that while the cascade centrality of i is sufficiently greater than j 's, it is not so high that it incentivizes agents to move into the $\sigma^* = (i, i)$ equilibrium. In fact, we can summarize this intuition with a clean sufficient condition.

Corollary 2. $\sigma^* = (i, j)$ (and $\sigma^* = (j, i)$ by symmetry) is a PSNE of Γ if for any $k \neq i, j$

¹⁰Note that there is a knife-edge case in which there are both types of equilibria.

- $C_i \geq \max \left\{ \frac{C_j}{2}, C_k \right\} + \epsilon(i, j)$
- $C_j \geq \max \left\{ \frac{C_i}{2}, C_k \right\} + \epsilon(j, i)$

Note the role that ϵ – the degree of interference between the agents – plays. ϵ is small when the agents are “far” from each other – in terms of the lengths and numbers of paths and the degrees of agents along these paths. If interference is small, then a sufficient condition for a PSNE with different seeds is that the cascade centrality of the agent with the highest cascade centrality in the network is no more than twice the cascade centrality of the second-highest agent.

Finally, Theorem 1 sheds light on why there is no PSNE in the social network in Figure 4. Condition 1 had no hope of being satisfied: there was no agent that has a far higher centrality than others. But Condition 2 could not be satisfied because there was a great deal of interference between the agents. There are several paths between i and j and C_k is very close to C_i and C_j making it difficult to satisfy any of the three parts of Condition 2.

4.3 Efficiency of equilibria: price of anarchy

We now turn to the efficiency properties of the PSNEs described above. It is easy to find an example of a social network in which none of the PSNEs is efficient (see Figure 7). We will focus on the price of anarchy, introduced by [18], which measures the ratio of the number of spreaders in the socially optimal outcome to the lowest expected number of spreaders among all Nash equilibria. The social planner’s objective is to maximize the sum of firms’ payoffs. Let $Y(\sigma) := \pi_A(\sigma) + \pi_B(\sigma)$ be this objective.

Definition 5. *Price of Anarchy of Γ is defined as:*

$$PoA(\Gamma) = \frac{\max_{\sigma \in \Sigma} Y(\sigma)}{\min_{\sigma \in \Sigma^*} Y(\sigma)}$$

The following theorem puts a bound on how “bad” equilibrium outcomes can be.

Theorem 2. *Consider a duopoly with unit budgets Γ . For any Γ that admits at least one PSNE,*

$$1 \leq PoA(\Gamma) < 1.5.$$

In fact, the (upper) bound is tight, which means we can generate a sequence of networks in which the supremum of the price of anarchy is equal to 1.5. This sequence consists of appending leaf edges to the same branch of a tree, starting with a star. Figure 5 illustrates two networks from this sequence. In both cases, the unique PSNE is one in which both firms seed agent i . Because the cascade centrality of i is so high, both firms are happy to take the gamble that involves getting a zero payoff. However, the social planner would prefer to seed i and j in both cases. It is easy to show that the expected number of spreaders in equilibrium is 5, whereas the social optimum approaches 7.5 as the length of the branch goes to infinity.

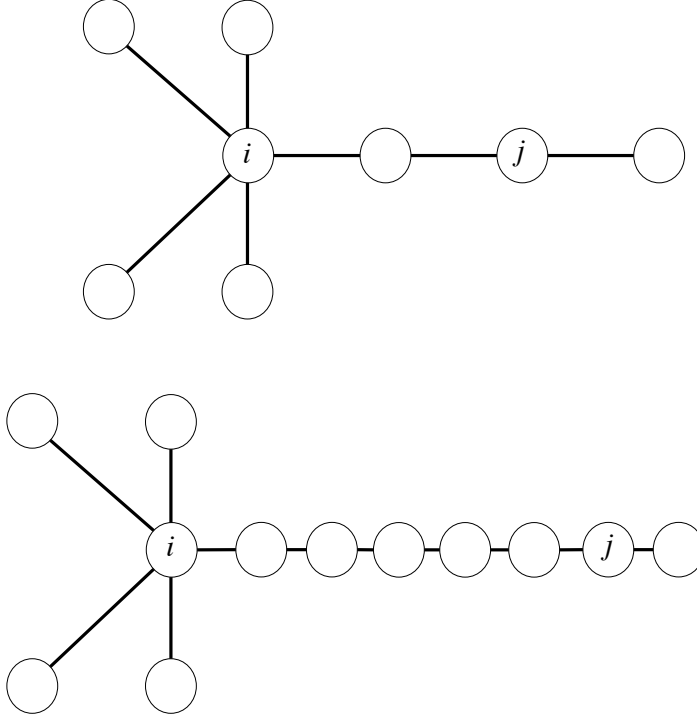


Figure 5: Sequence of networks that approaches the highest price of anarchy

4.4 Inequality of payoffs: budget multiplier

We consider the notion of the budget multiplier introduced by [12]. This metric measures the extent to which the network amplifies an asymmetry in budgets into an asymmetry in payoffs. Even though the budgets are equal in our model, equilibrium payoffs are not equal in general.

Definition 6. For arbitrary integer budgets \mathcal{B}_A and \mathcal{B}_B , the budget multiplier of game Γ is defined as:

$$BM(\Gamma) = \max_{\sigma \in \Sigma^*} \frac{\pi_A(\sigma)/\pi_B(\sigma)}{\mathcal{B}_A/\mathcal{B}_B}$$

So far, we have only analyzed the case of unit budgets, where $\mathcal{B}_A = \mathcal{B}_B = 1$. Hence, the budget multiplier in our case is simply $\max_{\sigma \in \Sigma^*} \frac{\pi_A(\sigma)}{\pi_B(\sigma)}$.

Theorem 3. For any Γ that admits at least one PSNE,

$$1 \leq BM(\Gamma) < 2$$

This (upper) bound is also tight. Consider the upper network in Figure 6. In this network, (i, j) is a unique PSNE (of course, (j, i) is also a PSNE by symmetry). Suppose that we create a sequence of networks, such that at every step of the sequence we increase the degree of i and j

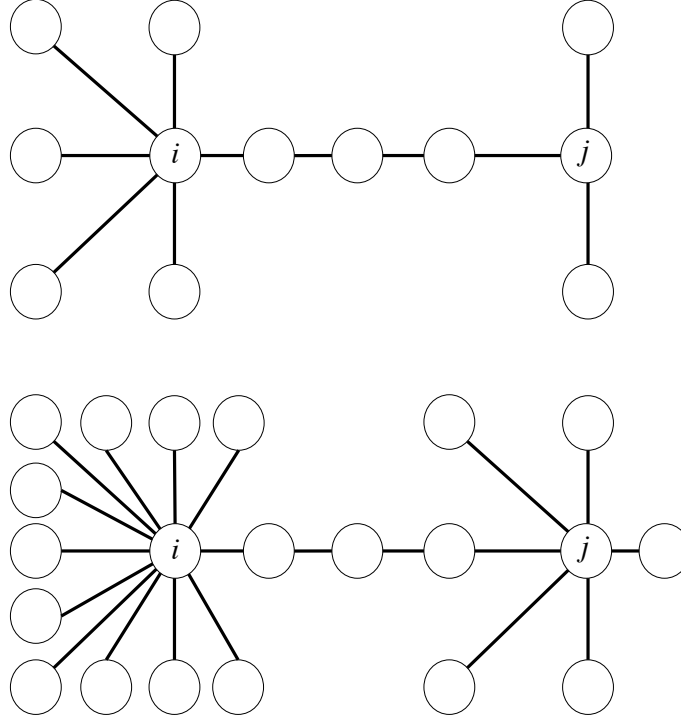


Figure 6: Sequence of networks that approaches the highest budget multiplier

by d_i and d_j respectively. This means that along the sequence the ratio d_i/d_j remains constant at 2 and the equilibrium remains unique (up to a symmetrical transformation). However, the limit payoff of the firm seeding at i is d_i , which is twice the limit payoff d_j of the firm seeding at j , as the degrees of these nodes increase along the sequence.

It is worth pointing out that the value of the upper bound of the budget multiplier is tied directly to the assumption that in the case when both firms seed the same node, the agent picks one rumor at random. However, even if the consumer is biased in his initial selection of the rumor, a (different) tight bound can still be found (see Section 5).

4.5 Competition on trees

We now illustrate the power of our approach by analyzing the duopoly on trees i.e. acyclic social networks. While empirically most social networks do not resemble trees, we hope to use them in order to crystallize the ideas developed in Section 3.

[19] show that cascade centrality of any agent in a tree is its degree plus one, which turns out to simplify our analysis substantially. In trees there is a unique path (and therefore a unique degree sequence product) between any two agents. Therefore, without loss of generality we can write $\mathcal{P}(i, j) = P(i, j)$. Denote by $\Delta_1(G)$ ($\Delta_2(G)$) the degree of the agent with the (weakly second-) highest degree in the network.

Definition 7. *Candidate sets of strategy profiles are defined as:*

- $\sigma_0 = \{(i, i) | d_i = \Delta_1\}$.
- $\sigma_1 = \{(i, j) | (d_i, d_j) \in \{(\Delta_1, \Delta_2), (\Delta_2, \Delta_1)\}\}$.
- $\sigma_2 = \{(i, j) | (d_i, d_j) \in \{(\Delta_1, \Delta_2 - 1), (\Delta_1 - 1, \Delta_2), (\Delta_2, \Delta_1 - 1), (\Delta_2 - 1, \Delta_1)\}\}$.

For each set of strategy profiles indexed by $l \in \{1, 2\}$ we can define the strategy profile that maximizes the degree sequence product among all strategy profiles in σ_l as $\sigma_l^* = \arg \max_{(i,j) \in \sigma_l} \chi_{P(i,j)}$. The value of this maximum degree sequence is denoted $\delta_l = \max_{(i,j) \in \sigma_l} \chi_{P(i,j)}$.

The following proposition characterizes all possible PSNEs on a tree in terms of the agents with the largest and second-largest degree and the “distance” (in terms of the degree sequence product) between them.

Proposition 2. *Suppose G is a tree and consider a duopoly with unit budgets Γ . Then, Γ admits a PSNE σ^* , which is characterized as follows:*

- If $\delta_1 = 1$, $\sigma^* = \begin{cases} \sigma_1^* & \text{if } \Delta_1 < 2\Delta_2 - 1 \\ \sigma_0 & \text{else if } \Delta_1 > 2\Delta_2 - 1 \\ \sigma_0 \cup \sigma_1^* & \text{otherwise} \end{cases}$
- Else if $\delta_1 = 2$, $\sigma^* = \begin{cases} \sigma_1^* & \text{if } \Delta_1/\Delta_2 < 2 \\ \sigma_0 & \text{else if } \Delta_1/\Delta_2 > 2 \\ \sigma_0 \cup \sigma_1^* & \text{otherwise} \end{cases}$
- Otherwise, $\sigma^* = \begin{cases} \sigma_1^* & \text{if } \Delta_1/\Delta_2 \leq 2 \\ \sigma_0 & \text{otherwise} \end{cases}$

Informally, Proposition 2 says that if the degree of the node with the largest degree ($\Delta_1(G)$) is more than double the degree of the node with the second-largest degree ($\Delta_2(G)$), then in the unique equilibrium both firms will seed the highest degree node. Otherwise, the equilibria will involve one firm seeding the largest-degree node and one firm seeding the second-largest-degree node. Let’s revisit our example from above. Figure 7 shows a network with both types of PSNEs: (j, j) , (j, m) and (j, i) (as well as symmetric counterparts of the latter two) because the largest degree (viz. 3) is exactly equal to twice the second-largest (viz. 2) minus one. Hence, the

We can also solve the social planner’s problem on trees explicitly.¹¹ Denote a solution by $\sigma^Y \in \arg \max_{\sigma \in \Sigma} \{Y(\sigma)\}$.

Proposition 3. *Suppose G is a tree. Then, if G is a star, $\sigma^Y = \sigma_0 \cup \sigma_1^*$. Otherwise,*

$$\sigma^Y = \begin{cases} \sigma_2^* & \text{if } \delta_1 = 1, \delta_2 > 2, \sigma_2^* \neq \emptyset \\ \sigma_1^* & \text{otherwise} \end{cases}$$

¹¹In general, this problem is NP-hard [15].

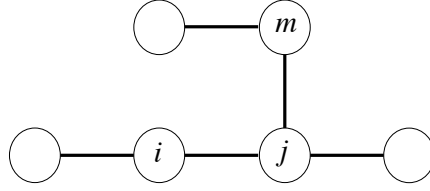


Figure 7: A tree with both types of PSNEs, neither of which are efficient

The network in Figure 7 satisfies the first condition of Proposition 3. None of the PSNEs in the network shown in Figure 7 are efficient. The socially optimal outcome is (i, m) and it involves seeding a node with the largest degree minus one and the second-largest-degree node.

Using Propositions 2 and 3, we can summarize necessary and sufficient conditions on the network structure that give rise to every possible combination of Nash equilibria.

Corollary 3. *Suppose G is a tree. Then, there*

- *are no efficient PSNEs iff $(\sigma^* = \sigma_0) \vee (\sigma^Y = \sigma_2^*)$*
- *are no inefficient PSNEs iff $(\sigma^* = \sigma_1^*) \wedge (\sigma^Y = \sigma_1^*)$*
- *is an inefficient PSNE iff $[(\sigma^* \supset \sigma_0) \wedge (\sigma^Y = \sigma_1^*)] \vee (\sigma^W = \sigma_2^*)$*
- *is an efficient PSNE iff $(\sigma^* \supset \sigma_1^*) \wedge (\sigma^Y = \sigma_1^*)$*
- *is an efficient and an inefficient PSNE iff $(\sigma^* = \sigma_0 \cup \sigma_1^*) \wedge (\sigma^Y = \sigma_1^*)$*

Any of these conditions relies on the computation of the inverse of the degree sequence product of a unique path between any pair of nodes (i, j) in a tree, where $i \neq j \in V$. One way to complete this task is:

- Compute the distance between any pair of nodes (i, j) where the distance $d(i, j)$ is initialized to:

$$d(i, j) = \begin{cases} 0 & \text{if } j = i \\ \log 1/d_j & \text{if } j \in N_i \\ \infty & \text{otherwise} \end{cases}$$

- For any $i \neq j \in V$, $\chi_{P(i, j)} = \exp(d(i, j))$ in this setup.

5 Extension: rumors of different persuasiveness

Thus far we have assumed that the rumors and the firms are completely symmetric. Now, suppose that rumour a is more persuasive than rumour b . We can adjust the diffusion dynamics to reflect this in two ways. First, we can assume that when both firms seed the same agent, he spreads rumour a with probability $\frac{1}{2} < \delta \leq 1$ and spreads rumour b with probability $1 - \gamma$. This change will only affect the initial distribution of seeds. Second, we can introduce further

stochasticity in the spreading behavior of rumor b . Once the spreading threshold is reached, agents would still spread rumor a with probability

$$\frac{\text{number of neighbors who spread } a \text{ at } t}{\text{total number of neighbors who spread either rumor at } t}$$

while we can change the probability of spreading b to

$$(1 - \gamma) \times \frac{\text{number of neighbors who spread } b \text{ at } t}{\text{total number of neighbors who spread either rumor at } t}$$

With the remaining probability the agent does not spread any rumor (despite reaching the threshold). Hence, rumor b will always be spread with a probability lower than what is implied by the proportion of neighbors who spread b , reflecting its inferior persuasiveness. We refer to the spreading process of rumor b as the “stochastic” linear threshold model in [19] and augment our definition of cascade centrality.

Proposition 4. *Stochastic cascade centrality of node i in graph G is the expected number of spreaders of rumor b in that graph given i is B ’s seed, namely*

$$\mathcal{C}_i(G, \gamma) = 1 + \sum_{j \in V \setminus \{i\}} \sum_{P \in \mathcal{P}_{ij}} \frac{(1 - \gamma)^{|P|}}{\chi_P}$$

where $|P|$ is the length of path P .

It is easy to see that once stochastic cascade centrality is defined, the firms’ profits functions are adjusted accordingly. All the necessary and sufficient conditions for the existence of PSNEs in general networks as well as sharp bounds for the budget multiplier and price of anarchy can be found analytically as before, with only minor modifications that reflect the role that γ and δ play in the diffusion process.

6 Conclusions

We introduced a tractable model of competitive rumor spread in social networks. Using the concept of cascade centrality, we have been able to fully characterize networks that admit PSNEs as well as many of their properties.

Many papers have looked at more general versions of this model by increasing the number of firms, allowing firms to spread more than one rumor and to have arbitrary budgets, or allowing for complementarities between rumors. However, we believe that our simple model not only provides sharper results, but also more analytical insight. There are plenty of interesting avenues for further work, such as studying mixed-strategy Nash equilibria, analyzing the existence of PSNEs in random graphs, looking at sequential rather than simultaneous entry, or considering other solution concepts, such as ϵ -Nash equilibria. Our equilibrium characterization could also be tested empirically using laboratory or field data.

Appendix

Proof of Proposition 1. There are two cases:

- When $\sigma_A = \sigma_B$, since the tie is broken with an equal probability for each firm, the problem can be casted as a single-seeding case with a seeding probability of a half. Hence, the payoff π_A takes the prescribed form.
- When $\sigma_A \neq \sigma_B$, for each firm, the problem can be analyzed as a multi-seeding single-firm case where the total number of seeds is 2. Note that since the selection decision is made using the latest spreader-neighbors only, the probability that a particular neighbor acts as a marginal influencer on node i is the same as under a multi-seed case (i.e. $\frac{1}{d_i}$). Therefore, the probability that a live-edge path [15] from a particular seed activates a node is same as under the multi-seed case. Suppose there are two seeds i, j . The probability that node $k \in V \setminus \{i, j\}$ spreads the rumour is

$$\sum_{P \in \mathcal{P}^*(i, k)} \frac{1}{\chi_P} + \sum_{P \in \mathcal{P}^*(j, k)} \frac{1}{\chi_P}$$

The probability that k is influenced by a live-edge path from seed i (of firm A) is $\sum_{P \in \mathcal{P}^*(i, k)} \frac{1}{\chi_P}$. But $\mathcal{P}(i, k) = \mathcal{P}(i\text{-via } j\text{-}k) \cup \mathcal{P}^*(i, k)$ (where, $\mathcal{P}^*(i, k)$ is defined as all paths from i to k not via the other seed j as in the main text), so probability that k is influenced by i can be written as

$$\sum_{P \in \mathcal{P}(i\text{-via } j\text{-}k)} \frac{1}{\chi_P} + \sum_{P \in \mathcal{P}^*(i, k)} \frac{1}{\chi_P}$$

Summing over all $k \in V$, gives us

$$\mathcal{C}_i = 1 + \sum_{k \in V} \sum_{P \in \mathcal{P}(i\text{-via } j\text{-}k)} \frac{1}{\chi_P} + \sum_{k \in V} \sum_{P \in \mathcal{P}^*(i, k)} \frac{1}{\chi_P}$$

Therefore,

$$1 + \sum_{k \in V} \sum_{P \in \mathcal{P}^*(i, k)} \frac{1}{\chi_P} = \mathcal{C}_i - \sum_{k \in V} \sum_{P \in \mathcal{P}(i\text{-via } j\text{-}k)} \frac{1}{\chi_P}$$

where $1 + \sum_{k \in V} \sum_{P \in \mathcal{P}^*(i, k)} \frac{1}{\chi_P} \equiv \pi_A(i, j)$ and $\sum_{k \in V} \sum_{P \in \mathcal{P}(i\text{-via } j\text{-}k)} \frac{1}{\chi_P} \equiv \epsilon(i, j)$ giving us the desired expression for the expected number of agents affected by seed i of firm A . The expression for seed j (of firm B) is analogous. □

Proof of Theorem 1. Fix a duopoly with unit budgets Γ . Then, there are two types of PSNE:

- Type 1: (i, i) for some $i \in V$.
- Type 2: (i, j) for some $i \neq j \in V$.

By Definition 4, for some $i \in V$, (i, i) is a type 1 PSNE if and only if

- $\pi_A(i, i) \geq \pi_A(j, i)$ for all $j \in V$,
 – $\pi_B(i, i) \geq \pi_B(i, j)$ for all $j \in V$.

By Proposition 1, these conditions hold if and only if $\frac{\mathcal{C}_i}{2} \geq \mathcal{C}_j - \epsilon(j, i)$, which is precisely the condition 1 upon re-writing.

Now, for some $i \neq j \in V$, (i, j) is a type 2 PSNE if and only if

- $\pi_A(i, j) \geq \pi_A(k, j)$ for all $k \in V$,
- $\pi_B(i, j) \geq \pi_B(i, k)$ for all $k \in V$.

By Proposition 1, these conditions hold if and only if

- $\mathcal{C}_i - \epsilon(i, j) \geq \mathcal{C}_k - \epsilon(k, j)$ for all $k \neq i, j$ and $\mathcal{C}_i - \epsilon(i, j) \geq \mathcal{C}_j/2$,
- $\mathcal{C}_j - \epsilon(j, i) \geq \mathcal{C}_k - \epsilon(k, i)$ for all $k \neq i, j$ and $\mathcal{C}_j - \epsilon(j, i) \geq \mathcal{C}_i/2$,

which is precisely the condition 2 upon re-writing. \square

Proof of Corollary 1. We apply a condition 1 from Theorem 1. Suppose $\mathcal{C}_i/\mathcal{C}_j \geq 2$ for any $j \neq i$. Then, $\mathcal{C}_i/\mathcal{C}_j \geq 2 \geq 2 - 2 \cdot \epsilon(j, i)/\mathcal{C}_j$ since $0 \leq \epsilon(j, i) \leq \mathcal{C}_j$. \square

Proof of Corollary 2. We apply a condition 2 from Theorem 1.

Suppose $\mathcal{C}_i \geq \max\{\mathcal{C}_j/2, \mathcal{C}_k\} + \epsilon(i, j)$. Then, $\mathcal{C}_i \geq \mathcal{C}_k + \epsilon(i, j) \geq \mathcal{C}_k + \epsilon(i, j) - \epsilon(k, j)$. This is the first condition upon dividing by \mathcal{C}_k . The second condition can be treated similarly.

Also, $\mathcal{C}_i \geq \max\{\mathcal{C}_j/2, \mathcal{C}_k\} + \epsilon(i, j)$ forces $\mathcal{C}_i \geq \mathcal{C}_j/2 + \epsilon(i, j)$, which is the first half of the third condition. The second half is analogous. \square

Proof of Theorem 2.

Lemma 1. Suppose (i, j) is a PSNE where $i \neq j \in V$. Then,

$$Y(i, j) \geq \frac{\mathcal{C}_i + \mathcal{C}_j}{2}.$$

Proof of Lemma 1. Since (i, j) is a PSNE, in particular,

$$\pi_A(i, j) \geq \pi_A(j, j) = \frac{1}{2} \cdot \mathcal{C}_j \text{ and } \pi_B(i, j) \geq \pi_B(i, i) = \pi_A(i, i) = \frac{1}{2} \cdot \mathcal{C}_i.$$

$$\text{Hence, } Y(i, j) = \pi_A(i, j) + \pi_B(i, j) \geq \frac{1}{2} \cdot (\mathcal{C}_i + \mathcal{C}_j).$$

\square

Lemma 2. For any $j \neq i$,

$$\epsilon(j, i) \leq \frac{1}{2} \mathcal{C}_j.$$

Proof of Lemma 2. Let $\delta(j, i) = \sum_{p \in \Lambda(j, i)} \frac{1}{\chi_p}$ where $\Lambda(j, i)$ is a set of all path starting from j that excludes i . Then, $\mathcal{C}_j = \epsilon(j, i) + \delta(j, i)$. So, the statement amounts to showing $\epsilon(j, i) \leq \delta(j, i)$.

Now, for each path $p \in \Xi(j, i)$, we may decompose it as $q - i - r$ where $q = (q_0, \dots, q_u) \in \Lambda(j, i)$ is a path with $q_j \neq i$ for all $j \in \{0, \dots, u\}$ and $r = (r_0, \dots, r_v)$ is a path with $r_0 \in N_i$.

Therefore, the statement follows from the following estimate:

$$\begin{aligned}
\epsilon(j, i) &= \sum_{p \in \Xi(j, i)} \frac{1}{\chi_p} = \sum_{p=q-i-r \in \Xi(j, i)} \frac{1}{\chi_p} \\
&\leq \sum_{q \in \Lambda(j, i)} \frac{1}{\chi_q} \cdot \frac{1}{d_i} \cdot \left(\sum_{r \text{ where } r_0 \in N_i - \{q_u\}} \frac{1}{\chi_r} \right) \\
&\leq \sum_{p \in \Lambda(j, i)} \frac{1}{\chi_p} \cdot \frac{\mathcal{C}_i - 1}{d_i} \\
&\leq \sum_{p \in \Lambda(j, i)} \frac{1}{\chi_p} \cdot \frac{(d_i + 1) - 1}{d_i} = \sum_{p \in \Lambda(j, i)} \frac{1}{\chi_p} = \delta(j, i).
\end{aligned}$$

□

Corollary 4. For any $i, j \in V$, $\pi_A(i, j) \geq \mathcal{C}_i/2$.

Proof of Corollary 4. This follows directly from Lemma 2 coupled with the formula for the payoff. □

Lemma 3. Suppose (i, j) is a social optimum and i, j , and k are pairwise distinct. Then,

$$\epsilon(i, k) - \epsilon(i, j) + \epsilon(j, k) - \epsilon(j, i) < \frac{1}{2}\mathcal{C}_k.$$

Proof of Lemma 3. For a fixed n , the graph configuration that minimizes $\frac{1}{2}\mathcal{C}_k - (\epsilon(i, k) - \epsilon(i, j) + \epsilon(j, k) - \epsilon(j, i))$ is a path of length n . It is easy to check for a path. □

Let σ^* be a PSNE, σ^Y be a social optimum, and ρ be the ratio $Y(\sigma^Y)/Y(\sigma^*)$. For any choice of σ^* and σ^Y , it suffices to show $\rho < 1.5$.

Suppose first that $\sigma^* = (i, i)$. There are few cases to consider depending on σ^Y . Indices i, j, k , and l are assumed to be pairwise distinct.

- 1. $\sigma^Y = (i, i)$. Then, $\rho = 1 < 1.5$.
- 2. $\sigma^Y = (i, j)$. Then, since (i, i) is a PSNE,

$$\pi_A(i, j) < \mathcal{C}_i \text{ and } \mathcal{C}_i/2 = \pi_B(i, i) \geq \pi_B(i, j).$$

Hence, $\rho < (\mathcal{C}_i + \mathcal{C}_i/2)/\mathcal{C}_i = 1.5$.

- 3. $\sigma^Y = (j, j)$. Then, since (i, i) is a PSNE, by Lemma 2,

$$\mathcal{C}_i/2 = \pi_A(i, i) \geq \pi_A(j, i) = \mathcal{C}_j - \epsilon(j, i) \geq \mathcal{C}_j/2.$$

Hence,

$$\rho = \frac{\mathcal{C}_j}{\mathcal{C}_i} \leq 1 < 1.5.$$

- 4. $\sigma^Y = (j, k)$. Then, since (i, i) is a PSNE, for any l , $\pi_A(l, i) \leq \pi_A(i, i) = \mathcal{C}_i/2$. So, $\mathcal{C}_l \leq \mathcal{C}_i/2 + \epsilon(l, i)$.

Hence,

$$\pi_A(j, k) = \mathcal{C}_j - \epsilon(j, k) \leq \frac{1}{2}\mathcal{C}_i + \epsilon(j, i) - \epsilon(j, k).$$

Similarly,

$$\pi_A(k, j) = \mathcal{C}_k - \epsilon(k, j) \leq \frac{1}{2}\mathcal{C}_i + \epsilon(k, i) - \epsilon(k, j).$$

Using Lemma 3,

$$\begin{aligned} \pi_A(j, k) + \pi_A(k, j) &\leq \left(\frac{1}{2}\mathcal{C}_i + \epsilon(j, i) - \epsilon(j, k) \right) + \left(\frac{1}{2}\mathcal{C}_i + \epsilon(k, i) - \epsilon(k, j) \right) \\ &\leq \mathcal{C}_i + (\epsilon(j, i) - \epsilon(j, k) + \epsilon(k, i) - \epsilon(k, j)) \\ &< \mathcal{C}_i + \frac{1}{2}\mathcal{C}_i = 1.5\mathcal{C}_i \end{aligned}$$

Therefore,

$$\rho = \frac{\pi_A(j, k) + \pi_A(k, j)}{\pi_A(i, i) + \pi_A(i, i)} = \frac{\pi_A(j, k) + \pi_A(k, j)}{\mathcal{C}_i} < \frac{1.5\mathcal{C}_i}{\mathcal{C}_i} = 1.5.$$

Suppose now $\sigma^* = (i, j)$. This time the only hard case is $\sigma^Y = (k, l)$; the four other cases can be treated using the similar argument as before.

Since (i, j) is a PSNE, for any t , $\pi_A(t, i) \leq \pi_A(j, i)$. In particular, $\mathcal{C}_k \leq \pi_A(j, i) + \epsilon(k, i)$. Hence,

$$\pi_A(k, l) = \mathcal{C}_k - \epsilon(k, l) \leq \pi_A(j, i) + \epsilon(k, i) - \epsilon(k, l).$$

Similarly,

$$\pi_A(l, k) = \mathcal{C}_l - \epsilon(l, k) \leq \pi_A(i, j) + \epsilon(l, i) - \epsilon(l, k).$$

Using Lemma 3,

$$\begin{aligned} \pi_A(k, l) + \pi_A(l, k) &\leq (\pi_A(j, i) + \epsilon(k, i) - \epsilon(k, l)) + (\pi_A(i, j) + \epsilon(l, i) - \epsilon(l, k)) \\ &\leq (\pi_A(i, j) + \pi_A(j, i)) + (\epsilon(k, i) - \epsilon(k, l) + \epsilon(l, i) - \epsilon(l, k)) \\ &< (\pi_A(i, j) + \pi_A(j, i)) + \frac{1}{2}\mathcal{C}_i \end{aligned}$$

Similarly,

$$\pi_A(k, l) + \pi_A(l, k) < (\pi_A(i, j) + \pi_A(j, i)) + \frac{1}{2}\mathcal{C}_j.$$

So, it follows that

$$\pi_A(k, l) + \pi_A(l, k) < (\pi_A(i, j) + \pi_A(j, i)) + \frac{1}{4}(\mathcal{C}_i + \mathcal{C}_j).$$

Therefore, by Lemma 1,

$$\begin{aligned} \rho &= \frac{\pi_A(k, l) + \pi_A(l, k)}{\pi_A(i, j) + \pi_A(j, i)} < \frac{(\pi_A(i, j) + \pi_A(j, i)) + (\mathcal{C}_i + \mathcal{C}_j)/4}{\pi_A(i, j) + \pi_A(j, i)} \\ &= \left(\frac{\pi_A(i, j) + \pi_A(j, i)}{\pi_A(i, j) + \pi_A(j, i)} \right) + \frac{1}{2} \cdot \left(\frac{(\mathcal{C}_i + \mathcal{C}_j)/2}{\pi_A(i, j) + \pi_A(j, i)} \right) \\ &\leq 1 + \frac{1}{2} = 1.5 \end{aligned}$$

Also, the bound is sharp since 1.5 bound can be achieved using the sequence of graphs shown in Figure 5. \square

Proof of Theorem 3. Suppose first that (i, i) is a PSNE. Then, the budget multiplier is $1 \leq 2$. So, suppose (i, j) is a PSNE with $i < j$. Then,

$$\pi_A(i, j) \geq \pi_A(j, j) = C_j/2 \text{ and } \pi_A(j, i) \geq \pi_A(i, i) = C_i/2.$$

Hence,

$$C_j/2 \leq \pi_A(i, j) < C_i \text{ and } C_i/2 \leq \pi_A(j, i) < C_j.$$

So,

$$\pi_A(i, j)/\pi_A(j, i) < 2 \text{ and } \pi_A(j, i)/\pi_A(i, j) < 2.$$

Therefore, the budget multiplier is

$$\max\left(\frac{\pi_A(i, j)}{\pi_A(j, i)}, \frac{\pi_A(j, i)}{\pi_A(i, j)}\right) < 2.$$

Also, the bound is sharp since 1.5 bound can be achieved using the sequence of graphs shown in Figure 6. \square

Proof of Proposition 2. Suppose (i, j) is a PSNE. Then, $\max(d_i, d_j) = \Delta_1$. Suppose $d_i, d_j < \Delta_1$. Then, there must be $k \neq j$ with $d_k = \Delta_1$ and $\pi_A(k, j) > \pi_A(i, j)$ since

$$\pi_A(i, j) = d_i + (1 - \chi_{P(i, j)}^{-1}) \leq (\Delta_1 - 1) + (1 - \chi_{P(i, j)}^{-1}) \quad (1)$$

$$= \Delta_1 - \chi_{P(i, j)}^{-1} < \Delta_1 \leq d_k + (1 - \chi_{P(k, j)}^{-1}) = \pi_A(k, j). \quad (2)$$

Also, $\min(d_i, d_j) \geq \Delta_2$. Without loss of generality, suppose $d_i < \Delta_2$ and $d_j = \Delta_1$. Then, using the similar argument to Equation 2, there is $k \neq j$ with $d_k = \Delta_2$ and $\pi_A(k, j) > \pi_A(i, j)$. Therefore, $(i, j) \in \sigma_0 \cup \sigma_1$.

Now, pick an element from σ_0 and σ_1 . Suppose that the chosen elements are $(1, 1) \in \sigma_0$ and $(2, 1) \in \sigma_1$ (with relabeling if necessary); for $(2, 1)$, suppose further that $\chi_{P(2, 1)} = \max_{(i, j) \in \sigma_1} \chi_{P(i, j)}$ with $d_2 = \Delta_2$ and $d_1 = \Delta_1$.

Lemma 4. Suppose $i \neq 1$. Then,

$$(a) \pi_A(2, 1) \geq \pi_A(i, 1) \text{ (b) and } \pi_B(2, 1) \geq \pi_B(2, j).$$

Proof of Lemma 4. For (a), the statement is trivial if $i = 2$. If $i \neq 2$ then $d_i \leq \Delta_3$. If $\Delta_3 = \Delta_2$ then the choice of $(2, 1)$ forces the inequality. Otherwise, the inequality follows from the similar argument to Equation (2).

For (b), the statement is trivial if $j = 1$. If $j = 2$ then,

$$\begin{aligned} 2\Delta_1 = \Delta_1 + \Delta_1 \geq \Delta_2 + 1 &\Rightarrow 2\Delta_1 + 2 \cdot (1 - \chi_{P(2, 1)}^{-1}) \geq \Delta_2 + 1 \\ \Leftrightarrow \pi_B(2, 1) = \Delta_1 + 1 - \chi_{P(2, 1)}^{-1} &\geq \frac{\Delta_2 + 1}{2} = \pi_B(2, 2) \end{aligned}$$

Otherwise, $d_j \leq \Delta_3$. So, the inequality follows from the similar argument that we used to prove (a). \square

By Lemma 4, all PSNE's involve firms seeding at highest and second-highest nodes; that is, it suffices to compare $\pi_A(1, 1)$ and $\pi_A(2, 1)$ to classify all PSNE's, where $\pi_A(1, 1) = \frac{\Delta_1 + 1}{2}$ and $\pi_A(2, 1) = \Delta_2 + 1 - \chi_{P(2,1)}^{-1}$.

- If $\pi_A(1, 1) = \pi_A(2, 1)$ then both $(1, 1)$ and $(2, 1)$ are PSNE's.
 - Else if $\pi_A(1, 1) > \pi_A(2, 1)$ then $(1, 1)$ is a PSNE.
 - Otherwise, $(2, 1)$ is a PSNE.

When $\chi_{P(2,1)} = 1$, we need to compare:

$$\frac{\Delta_1 + 1}{2} \text{ and } \Delta_2,$$

or, equivalently,

$$\Delta_1 \text{ and } 2\Delta_2 - 1.$$

If $\Delta_1 > 2\Delta_2 - 1$, $(1, 1)$ is a PSNE. If $\Delta_1 < 2\Delta_2 - 1$ then $(2, 1)$ is a PSNE. If $\Delta_1 = 2\Delta_2 - 1$ both $(1, 1)$ and $(2, 1)$ are PSNE's.

Now, when $\chi_{P(2,1)} = 2$, we need to compare:

$$\frac{\Delta_1 + 1}{2} \text{ and } \Delta_2 + \frac{1}{2},$$

or, equivalently,

$$\Delta_1 \text{ and } 2\Delta_2.$$

Finally, when $\chi_{P(2,1)} > 2$, we need to compare:

$$\frac{\Delta_1 + 1}{2} \text{ and } \Delta_2 + 1 - \chi_{P(2,1)}^{-1}$$

or, equivalently,

$$\Delta_1 \text{ and } 2\Delta_2 + \delta$$

where $\delta := 1 - 2\chi_{P(2,1)}^{-1}$. Note that $\delta \in (0, 1)$. In particular, Δ_1 cannot be equal to $2\Delta_2 + \delta$ since $\Delta_1, \Delta_2 \in \mathbb{N}$.

Finally, note that $\Delta_1 > 2\Delta_2 + \delta$ if and only if $\Delta_1 \geq 2\Delta_2 + 1$ since $\Delta_1, \Delta_2 \in \mathbb{N}$. Similarly, $\Delta_1 < 2\Delta_2 + \delta$ if and only if $\Delta_1 \leq 2\Delta_2$. \square

Proof of Proposition 3. Suppose $x \in \sigma_1^*$. Then, with relabeling if necessary, x is $(1, 2)$ where $d_1 = \Delta_1$ and $d_2 = \Delta_2$. Now, pick $y = (i, j)$ where $i, j \in V$ and $d_i \geq d_j$. Let $\Delta_x = \Delta_1 + \Delta_2$ and $\Delta_y = d_i + d_j$.

Suppose $Y(y) > Y(x)$. Then, $i \neq j$. If $i = j$ then $Y(y) = d_i + 1$, which forces

$$Y(x) = \Delta_1 + \Delta_2 + 2 \cdot (1 - \chi_{P(1,2)}^{-1}) \geq \Delta_1 + \Delta_2 \geq d_i + \Delta_2 \geq d_i + 1 = Y(y).$$

Suppose $Y(y) = Y(x)$. If $i = j$ then

$$\Delta_1 + \Delta_2 + 2 \cdot (1 - \chi_{P(1,2)}^{-1}) = Y(x) = Y(y) = d_i + 1$$

Since $d_i + 1 \in \mathbb{N}$, $Y(x)$ is either $\Delta_x + 1$ or Δ_x because $\chi_{P(1,2)}^{-1} \in (0, 1)$. Suppose first that $Y(x) = \Delta_x + 1$. Then, $\Delta_1 + \Delta_2 - 1 = d_i \leq \Delta_1$ forces $\Delta_2 \leq 1$. Since $\Delta_2 \geq 1$, $\Delta_2 = 1$. Thus, $d_i = \Delta_1$ and $\Delta_2 = 1$; that is, the network has to be a star. On a star, $\sigma^Y = \sigma_0 \cup \sigma_1^*$. Now, suppose $Y(x) = \Delta_x$. Then, $\Delta_1 + \Delta_2 = d_i \leq \Delta_1$. Hence, $\Delta_2 \leq 0$. But, since $\Delta_2 \geq 1$, this is clearly a contradiction.

Note that $\Delta_x - \Delta_y \geq 0$. First, suppose $\Delta_x - \Delta_y \geq 2$. Then,

$$Y(y) = d_i + d_j + 2 \cdot (1 - \chi_{P(i,j)}^{-1}) < d_1 + d_2 + 2 \cdot (1 - \chi_{P(1,2)}^{-1}) = Y(x)$$

if and only if

$$2\chi_{P(1,2)}^{-1} - 2\chi_{P(i,j)}^{-1} < \Delta_x - \Delta_y,$$

which holds since

$$2\chi_{P(1,2)}^{-1} - 2\chi_{P(i,j)}^{-1} < 2\chi_{P(1,2)}^{-1} \leq 2 \leq \Delta_x - \Delta_y.$$

Now, suppose $\Delta_x - \Delta_y = 0$. Then, necessarily $d_i = \Delta_1$ and $d_j = \Delta_2$. Hence, $Y(y) \leq Y(x)$ by the choice of x .

Finally, for the case $\Delta_x - \Delta_y = 1$, first suppose $2 \notin N_1$. Then, $\chi_{P(1,2)} \geq 2$, or $\chi_{P(1,2)}^{-1} \leq 1/2$. Hence,

$$2\chi_{P(1,2)}^{-1} - 2\chi_{P(i,j)}^{-1} < 2\chi_{P(1,2)}^{-1} \leq 1 \leq \Delta_x - \Delta_y.$$

Rearranging this inequality yields

$$Y(x) = \Delta_x + 2 \cdot (1 - \chi_{P(1,2)}^{-1}) > \Delta_y + 2 \cdot (1 - \chi_{P(i,j)}^{-1}) = Y(y).$$

Finally, suppose $2 \in N_1$. Then,

$$\begin{aligned} Y(y) > Y(x) &\Leftrightarrow \Delta_y + 2 \cdot (1 - \chi_{P(i,j)}^{-1}) > \Delta_x \\ &\Leftrightarrow 2\chi_{P(i,j)}^{-1} < 2 - (\Delta_x - \Delta_y) = 1 \\ &\Leftrightarrow \chi_{P(i,j)} > 2 \end{aligned}$$

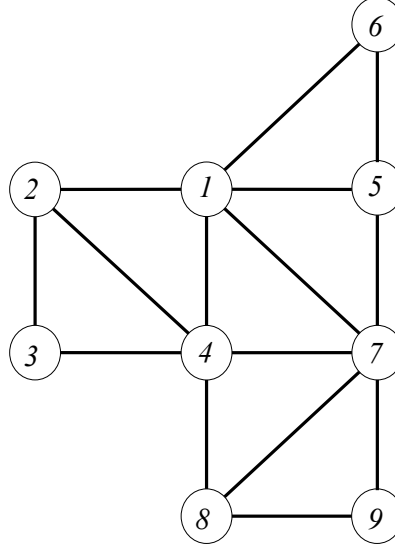
Hence, if $\delta_1 = 1$, $\delta_2 > 2$, and $\sigma_2^* \neq \emptyset$ then $\sigma^Y = \sigma_2^*$. Otherwise, $\sigma^Y = \sigma_1^*$ unless G is a star; for a star, $\sigma^Y = \sigma_0 \cup \sigma_1^*$. \square

Proof of Corollary 3. Note that a PSNE σ^* is efficient if and only if it is also social optimum. So, for each of the conditions, it suffices to check explicit characterizations of the equilibria and social optima from Proposition 2 and Proposition 3. \square

Proof of Figure 4. $\pi_A(i, j)$ can be computed as:

$$\begin{pmatrix} 2.287 & 3.625 & 4.058 & \mathbf{3.417} & \mathbf{4.033} & \mathbf{4.167} & 3.383 & 3.470 & 3.594 \\ 2.265 & 1.608 & 2.290 & 2.083 & 2.854 & 2.999 & 2.693 & 2.874 & 2.930 \\ 1.991 & 1.594 & 1.241 & 1.528 & 2.206 & 2.325 & 2.122 & 2.283 & 2.323 \\ 3.383 & 3.470 & 3.594 & 2.287 & 3.625 & 4.058 & \mathbf{3.417} & \mathbf{4.033} & \mathbf{4.167} \\ 2.693 & 2.874 & 2.930 & 2.265 & 1.608 & 2.290 & 2.083 & 2.854 & 2.999 \\ 2.122 & 2.283 & 2.323 & 1.991 & 1.594 & 1.241 & 1.528 & 2.206 & 2.325 \\ \mathbf{3.417} & \mathbf{4.033} & \mathbf{4.167} & 3.383 & 3.470 & 3.594 & 2.287 & 3.625 & 4.058 \\ 2.083 & 2.854 & 2.999 & 2.693 & 2.874 & 2.930 & 2.265 & 1.608 & 2.290 \\ 1.528 & 2.206 & 2.325 & 2.122 & 2.283 & 2.323 & 1.991 & 1.594 & 1.241 \end{pmatrix}$$

Figure 8: Example of a social network that does not admit a PSNE



In particular, we note that $\pi(7, 1)$ and $\pi(4, 1)$ are different. We show this by an explicit calculation using counting formula. We assume that the seed of firm B is always fixed at 1. Let's perform a DFS (depth-first search) from a seed of firm A in a lexicographic order. There are 20 distinct paths in the graph emanating from 4 that avoids 1 as shown in Table 1.

It is immediate from here that:

$$\begin{aligned}\pi(4, 1) &= 1 + \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{5} \cdot 1 + \frac{1}{6} \cdot 3 + \frac{1}{10} \cdot 1 + \frac{1}{15} \cdot 3 + \frac{1}{30} \cdot 5 \\ &\quad + \frac{1}{45} \cdot 1 + \frac{1}{90} \cdot 2 + \frac{1}{180} \cdot 1 \\ &= \frac{203}{60} \approx 3.383\end{aligned}$$

There are 23 distinct paths in the graph emanating from 7 that avoids 1 as shown in Table 2. It is immediate from here that:

$$\begin{aligned}\pi(7, 1) &= 1 + \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{5} \cdot 1 + \frac{1}{6} \cdot 3 + \frac{1}{10} \cdot 1 + \frac{1}{15} \cdot 3 + \frac{1}{30} \cdot 5 \\ &\quad + \frac{1}{45} \cdot 1 + \frac{1}{60} \cdot 1 + \frac{1}{90} \cdot 3 + \frac{1}{180} \cdot 2 \\ &= \frac{41}{12} \left(= \frac{205}{60} \right) \approx 3.417\end{aligned}$$

So, the difference of the payoffs is precisely:

$$\pi(7, 1) - \pi(4, 1) = \frac{1}{60} \cdot 1 + \frac{1}{90} \cdot 1 + \frac{1}{180} \cdot 1 = \frac{1}{30}$$

Now, it is clear from here that when 7 is the best response for 1, 4 is for 1, and 1 is for 4 by symmetry. Therefore, we conclude that there cannot be a PSNE in Figure 4. \square

Proof of Figure 7. Here are all PSNE's:

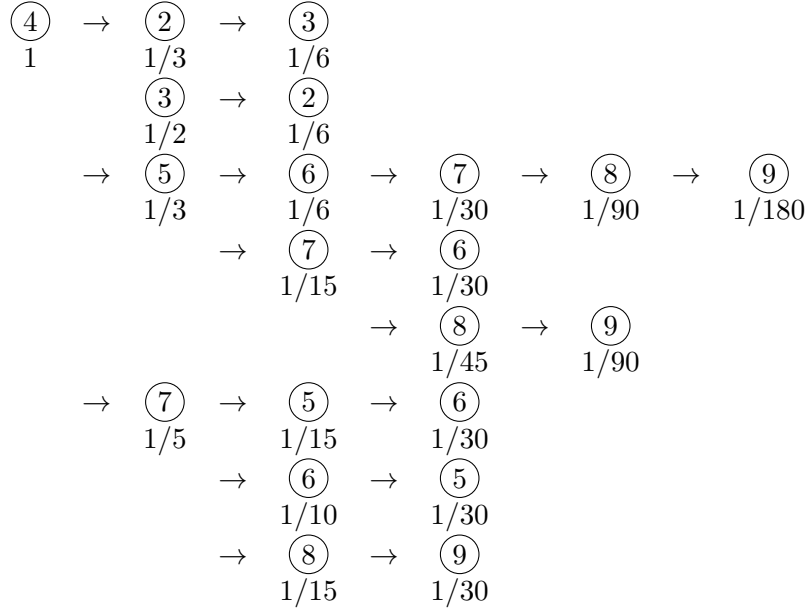


Table 1: $\pi(4, 1)$

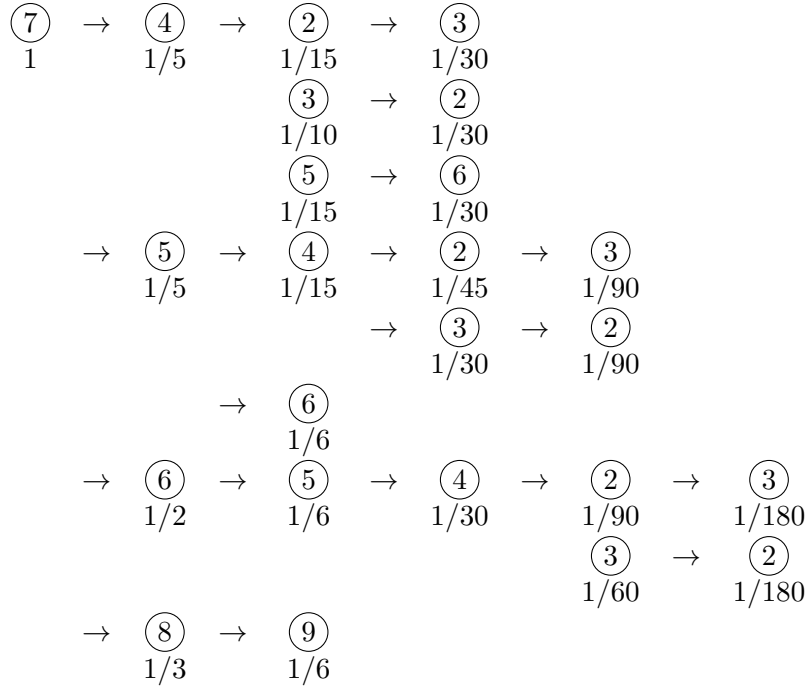


Table 2: $\pi(7, 1)$

- $\sigma_1^* = (i, j)$ and $\sigma_{1'}^* = (j, i)$ with $Y(\sigma_1^*) = 5$.
 - $\sigma_2^* = (i, k); \sigma_{2'}^* = (k, i)$ with $Y(\sigma_2^*) = 5$.
 - $\sigma_3^* = (i, i)$ with $Y(\sigma_3^*) = 4$.

Here is the unique social optimal solution:

- $\sigma_1^Y = (j, k)$ and $\sigma_{1'}^Y = (k, j)$ with $Y(\sigma_1^Y) = 16/3$.

In particular, note that none of the PSNE's is efficient since $Y(\sigma_i^*) < Y(\sigma_j^Y)$ for any valid pair (i, j) . □

Proof of Proposition 4. Follows directly from Proof of Proposition 1. □

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