Gross substitutes and complements: a simple generalization

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Abstract

This paper extends the gross substitutes and complements (GSC) framework of Sun and Yang (2006) to a more general substitutes and complements structure. We show that competitive equilibrium with indivisible goods exists under significantly weaker, easily checkable, and interpretable conditions. We show that the dynamic double-track procedure developed by Sun and Yang (2008, 2009) can be extended to find the competitive equilibrium outcome. We also apply these insights to trading networks.

Keywords: indivisibility, competitive equilibrium, substitutes, complements, auctions.

JEL Classification: D44, D47, D51

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1 Introduction

In this paper, we study an exchange economy in which agents have heterogeneous preferences over indivisible items. The relationship between such exchange economies, auctions, and matching markets is already well known for the case of substitutable goods (Kelso and Crawford, 1982, Milgrom, 2000, Milgrom and Strulovici, 2009). However, in labor markets, may view skills of different workers are complementary. For example, a hospital may want to hire a surgeon together with an anaesthetist. Technological complementarities also occur in many designed markets, such as telecommunications auctions. For instance, in the Japanese 4G spectrum auction, there were 10 lots of 20MHz spectrum bands. There were two competing technologies: TDD and FDD. The FDD technology required paired lots – uplink and a downlink – which had to be located sufficiently far away from each other on the spectrum. For a firm wanting to deploy FDD, any (potential) uplink or any downlink bandwidth is substitutable (hence bundling is not trivial), but an uplink and a downlink band are complementary. On the other hand, TDD only required any one of (or several adjacent) substitutable spectrum band lots (Matsushima, 2012). However, it is well known that, in markets with indivisible commodities, competitive equilibrium does not always exist when complementarities are present (Kelso and Crawford, 1982, Gul and Stacchetti, 1999). This paper offers a new sufficient condition (satisfied for the Japanese spectrum auction) for the existence of competitive equilibrium in an exchange economy in which agents trade indivisible substitute and complement goods.

This paper builds on the gross substitutes and complements (GSC) preference framework introduced by Sun and Yang (2006). They showed that if, for example, a seller offers trousers and shirts and all buyers regard any two shirts (or any two pairs of trousers) as substitutes, but any shirt and pair of trousers as complements, then competitive equilibrium will exist in this economy when agents’ utility functions are quasilinear in prices. In the present model, all goods can be partitioned into sets of substitutes and every buyer regards goods from some two partition elements as complements. As an example, consider an economy in which the seller offers three types of goods: jackets, trousers, and shirts. Buyers view any type of good as substitutes – this is a natural assumption when the goods of a particular type are sufficiently similar. There are also two types of buyer: a student who views jackets and trousers as complements, and a professor who views trousers and shirts as complements. We show that in this sort of economy competitive equilibrium is guaranteed to exist. However, if we add another agent into the economy – a post-doc who regards jackets and shirts as complements – then competitive equilibrium is no longer guaranteed to exist (see Section 2.3). This failure of equilibrium existence occurs because there is an odd cycle in
the \textit{generalized gross substitutes and complements} (GGSC) structure of agents’ preferences: jackets and trousers are complements for the student, trousers and shirts are complements for the professor, and shirts and jackets are complements for the post-doc. In Section 3, we show that competitive equilibrium exists whenever these odd cycles are absent: a much weaker, yet intuitive, condition than those found previously.\footnote{The “no odd party” (Tan, 1991) and “no-odd-rings” conditions (Chung, 2000) guarantee existence of stable matchings in the roommate market. Gudmundsson (2013) showed that absence of odd cycles in a certain linear programming problem guarantees existence of equilibrium in the partnership formation problem (Talman and Yang, 2011). However, these results for one-sided matching problems are logically unrelated to the present chapter.} Other generalizations of the GSC framework were proposed by Baldwin and Klemperer (2013) and Shioura and Yang (2013), but only the latter developed an auction procedure.

We proceed as follows. The model as well as some motivating examples are described in Section 2 and the main result is stated and proved in Section 3. In Section 4, we show that an ingenious tâtonnement procedure developed by Sun and Yang (2009) can be easily extended to find the competitive equilibrium allocation and prices in our economy. In Section 5, we show how to apply our insights to trading networks (Hatfield et al., 2013, Drexl, 2013) and multi-unit exchange economies (Shioura and Yang, 2013).

\section{Model}

\subsection{Ingredients}

There is a finite set of agents \(i \in I\) and a finite set of indivisible goods \(\omega \in \Omega\) in the economy. Goods are partitioned into \(M\) (possibly empty) disjoint subsets (of similar goods) of \(\Omega\), forming a set \(S = \{S_1, \ldots, S_M\}\) such that \(S_n \cap S_m = \emptyset\) (where \(n, m = 1, \ldots, M; n \neq m\)) and \(\bigcup_{m=1}^{M} S_m = \Omega\). Each element of the partition represents a set of similar goods (such as shirts of a different color). Let \(\Psi \in 2^\Omega\) be a bundle of goods and \(\Psi_i\) be a bundle for agent \(i \in I\). Denote \(p_\omega\) as the price of good \(\omega\) and \(p \in \mathbb{R}^{|\Omega|}\) as the price vector. An allocation is a partition \(\Pi\) of goods into (possibly empty) bundles for different agents (\(\Pi = \{\Psi_i\}_{i \in I}\) such that \(\bigcup_{i \in I} \Psi_i = \Omega\) and \(\Psi_i \cap \Psi_j = \emptyset\)). An arrangement is a pair \([\Pi, p]\), which associates prices to goods in the economy. We assume that agents’ net utility functions are quasilinear in prices

\[ U_i([\Pi; p]) \equiv u_i(\Psi_i) - \sum_{\omega \in \Psi_i} p_\omega \tag{1} \]

where \(u_i : 2^\Omega \rightarrow \mathbb{R}_+\) is the valuation function with \(u_i(\emptyset) = 0\) and agents are not subject to any liquidity or budget constraints.\footnote{These assumptions are necessary for the results and relaxing them is an interesting open problem.} Therefore, the trading economy can be described
by the set of goods and the agents’ valuations of every bundle: \( \mathcal{E} \equiv \{ \Omega, (u_i, i \in I) \} \). The demand correspondence \( D_i : \mathbb{R}^{|\Omega|} \to 2^\Omega \) for agent \( i \) is defined in the usual way

\[
D_i(p) \equiv \arg \max_{\Psi \subseteq \Omega} U_i([\Psi; p])
\]

A competitive equilibrium in this economy consists of an allocation of goods to agents and a vector of competitive prices, such that the market clears: every agent demands precisely his allocation at this price vector.

Definition 1. Competitive equilibrium is an arrangement \([\Pi; p]\) such that for all \( i \in I \), \( \Psi_i \in D_i(p) \).

2.2 Preferences

First, we formally define preferences that satisfy gross substitutes and complements (Sun and Yang, 2006). Let us consider \( S = S^* = \{ S_1, S_2 \} \), which represents shirts \( S_1 \) and trousers \( S_2 \). For notation purposes, define \( e(k) \) is a \( k \)th unit vector in \( \mathbb{R}^{|\Omega|} \) and \( A^c = \Omega \setminus A \).

Definition 2 (Sun and Yang, 2006). Preferences of agent \( i \) satisfy gross substitutes and complements (GSC) on \( S^* \) if for any \( p \in \mathbb{R}^{|\Omega|} \), \( \omega_k \in S_m \), \( \delta \geq 0 \), \( A \in D_i(p) \), there exists \( B \in D_i(p + \delta e(k)) \) such that \( [A \cap S_m] \setminus \{ \omega_k \} \subseteq B \) and \( A^c \cap S_m^c \subseteq B^c \).

In other words, preferences of agent \( i \) satisfy GSC if whenever the price of a shirt (trousers) increases, \( i \)’s resulting demand, \( B \), for other shirts (trousers) does not fall i.e \( [A \cap S_m] \setminus \{ \omega_k \} \subseteq B \), and demand for trousers (shirts) does not rise i.e. \( A^c \cap S_m^c \subseteq B^c \). Using our example in the Introduction, the professor has GSC preferences over shirts and trousers. Sun and Yang (2006) show that whenever preferences of all agents satisfy GSC competitive equilibrium exists. We now turn to the main assumption on individual preferences and structure of the economy which generalizes GSC.

Definition 3. Preferences satisfy generalized gross substitutes and complements (GGSC) on \( S \) if for any \( p \in \mathbb{R}^{|\Omega|} \), \( \omega_k \in S_m \), \( \delta \geq 0 \), \( A \in D_i(p) \), \( S_m \in S \) and \( i \in I \), there exists one \( S_n \in S \) and \( B \in D_i(p + \delta e(k)) \) such that \( [A \cap S_m] \setminus \{ \omega_k \} \subseteq B \), \( A^c \cap S_n \subseteq B^c \) and \( A \cap [S_m \cup S_n]^c = B \cap [S_m \cup S_n]^c \).

In words, agents’ demand correspondences have GGSC structure if we can divide goods into a partition \( S \) (for all agents) such that, whenever we consider preferences over goods contained in any two elements of \( S \) in isolation, these preferences satisfy GSC for some agents. From now on, whenever we say that agents have GSC preferences over \( S_m \) and \( S_n \),
we will mean that these agents would have GSC preferences over \( S^* = \{S_m, S_n\} \) if the goods in \( S_m \) and \( S_n \) were considered in isolation. Different agents may have GSC preferences over different pairs of the partition elements of \( S \).\(^3\) Again returning to our example from the Introduction, the student has GSC preferences over jackets and trousers and the post-doc has GSC preferences over jackets and shirts: so the agents’ preferences satisfy GGSC. It is worth noting that, for agents who have GSC preferences over \( S_m \) and \( S_n \), changes in prices for a good contained in \( S_m \) do not have any effect on the demands for goods outside \( S_n \cup S_m \), and changes in prices for goods contained \([S_n \cup S_m]^c\) do not affect the demands for goods in \( S_n \cup S_m \). In other words, for these agents, goods in \( S_n \cup S_m \) and \([S_n \cup S_m]^c\) are independent. To see how the GGSC structure generalizes GSC, note that preferences with a GGSC structure satisfy GSC for all agents if \( M = 2 \). Since we do not rule out that \( S = \emptyset \), it is clear that if \( M = 2 \) and \( S = \{S, \emptyset\} \), we return to the gross substitutes framework of Kelso and Crawford (1982).\(^4\)

### 2.3 Motivating examples

Will a competitive equilibrium exist in a trading economy where agents’ preferences satisfy GGSC?

**Example 1**

Consider a trading economy with four buyers \( i, j, k, l \), a seller \( s \), and four goods \( \{\omega_1, \omega_2, \omega_3, \omega_4\} \). The seller’s values are zero for every bundle. Buyers get utility 1 from getting a bundle containing their two desired goods (the bundle may include other goods) and zero otherwise as illustrated in Table 1. Agent \( i \) has GSC preferences over \( \{\omega_1, \omega_2\} \); \( j \) over \( \{\omega_2, \omega_3\} \); \( k \) over \( \{\omega_3, \omega_4\} \); and \( l \) over \( \{\omega_4, \omega_1\} \). Hence, \( S_m = \{w_m\} \) for \( m = 1, \ldots, 4 \) and preferences satisfy GGSC. Suppose that the price vector is \( p^* = (p_1, p_2, p_3, p_4) = (1, 0, 1, 0) \) and \( \Pi^* = \{\Psi_i, \Psi_j, \Psi_k, \Psi_l\} = \{\{\omega_1, \omega_2\}, \emptyset, \{\omega_3, \omega_4\}, \emptyset\} \). Bundle \( \{\omega_1, \omega_2\} \) is allocated to agent \( i \) and \( k \) gets bundle \( \{\omega_3, \omega_4\} \); agents \( j \) and \( l \) get nothing. Arrangement \([\Pi^*; p^*]\) constitutes a competitive equilibrium.

How do we find this competitive equilibrium? Suppose, a seller announces an initial price vector \( p(0) = (2, 0, 2, 0) \) i.e. selecting high prices for \( \omega_1 \) and \( \omega_3 \) and low prices for \( \omega_2 \) and \( \omega_4 \). Now, \( \omega_1 \) and \( \omega_3 \) are under-demanded and \( \omega_2 \) and \( \omega_4 \) are balanced (according to the auction

\(^3\)We could allow the same agent to have GSC preferences over several pairs of elements of \( S \) (because of the quasilinearity of the utility functions), but, while this complicates the exposition, it does not affect the results in any way.

\(^4\)Therefore, we do not rule out that some agents may only view goods within an element \( S \) as substitutes and no goods as complements.
Table 1: Competitive equilibrium with no odd cycles in the GGSC structure

<table>
<thead>
<tr>
<th>$\Psi$</th>
<th>$u_i(\Psi_i)$</th>
<th>$u_j(\Psi_j)$</th>
<th>$u_k(\Psi_k)$</th>
<th>$u_l(\Psi_l)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$ and otherwise</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>${\omega_1, \omega_2}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>${\omega_2, \omega_3}$</td>
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<td>0</td>
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<tr>
<td>${\omega_3, \omega_4}$</td>
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<td>1</td>
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<tr>
<td>${\omega_1, \omega_2, \omega_3}$</td>
<td>1</td>
<td>1</td>
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<td>${\omega_2, \omega_3, \omega_4}$</td>
<td>0</td>
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<td>${\omega_1, \omega_2, \omega_3, \omega_4}$</td>
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</table>

Table 2: No competitive equilibrium with an odd cycle in the GGSC structure

<table>
<thead>
<tr>
<th>$\Psi$</th>
<th>$u_i(\Psi_i)$</th>
<th>$u_j(\Psi_j)$</th>
<th>$u_k(\Psi_k)$</th>
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</thead>
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<tr>
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<td>0</td>
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<tr>
<td>${\omega_1, \omega_2}$</td>
<td>1</td>
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<td>${\omega_2, \omega_3}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>${\omega_3, \omega_1}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>${\omega_1, \omega_2, \omega_3}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

rules). Therefore, the seller reduces the prices of $\omega_1$ and $\omega_3$ and keeps the prices of $\omega_2$ and $\omega_4$ constant. At $p(1) = (1, 0, 1, 0)$, all goods are balanced, so this price vector constitutes a competitive equilibrium.

Example 2

This example illustrates that competitive equilibrium does not always exist when agents’ preferences satisfy GGSC. This example is adapted from Bikhchandani and Mamer (1997).

Consider three buyers $i, j, k$, a seller $s$, and three goods $\omega_1, \omega_2, \omega_3$. The seller’s values are zero for every bundle. Agents get utility 1 from getting a bundle containing their two desired goods (the bundle may include other goods) and zero otherwise as illustrated in Table 2. Once again, $S_m = \{w_m\}$ for $m = 1, \ldots, 3$ and preferences satisfy GGSC. The problem is
symmetric so without loss of generality assume that in a competitive equilibrium allocation \( \Psi_i = \{\omega_1, \omega_2, \omega_3\} \).\(^5\) Then, it must be the case that

\[
\begin{align*}
    p_1 + p_2 + p_3 &\leq 1 \\
    p_2 + p_3 &\geq 1 \\
    p_3 + p_1 &\geq 1
\end{align*}
\] (3)

otherwise agent \( i \) would not demand \( \{\omega_1, \omega_2, \omega_3\} \) and agents \( j \) or \( k \) would demand a bundle. However, adding equations (4) and (5) and using equation (3), we obtain

\[
\begin{align*}
    p_1 + p_2 + 2p_3 &\geq 2 \\
    2 - 2p_3 &\leq p_1 + p_2 \leq 1 - p_3
\end{align*}
\] (6)

which only holds when \( p_3 = 1 \) implying that \( p_1 + p_2 \leq 0 \). Hence, the seller would not sell the bundle and this allocation cannot be supported by market clearing prices.\(^6\)

2.4 GGSC cycles

We now introduce a formal definition of an odd GGSC cycle, which was described in the Introduction.

**Definition 4.** Preferences form a *GGSC cycle* on \( S \) if

1. Preferences satisfy GGSC on \( S \), and
2. Non-empty elements of \( S \) can be arranged in some order \( S_1 \ldots S_K \) (\( 2 \leq K \leq M \)) such that there exists an agent with GSC preferences over goods contained in \( S_m \) and \( S_{m+1} \) for \( m = 1, \ldots, K \) (\( m \mod K \)).

Preferences may form GGSC cycles of different lengths. In Sun and Yang’s framework, the largest GGSC cycle is of length 2. In Example 1, we saw a GGSC cycle of size 4. We want to focus on the existence of GGSC cycles of odd length.

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\(^5\)We only need to consider efficient allocations.

\(^6\)Another candidate is for a competitive equilibrium allocation is, without loss of generality, \( \Psi_i = \{\omega_1, \omega_2\} \), leaving \( \omega_3 \) unsold. This implies \( p_3 = 0 \). However, the following inequalities must hold

\[
\begin{align*}
    p_1 + p_2 &\leq 1 \\
    p_2 + p_3 &\geq 1 \\
    p_3 + p_1 &\geq 1
\end{align*}
\]

but this implies that \( p_1 \geq 1, p_2 \geq 1, \) and \( 2 \leq p_1 + p_2 \leq 1 \), which means that this allocation cannot be supported in a competitive equilibrium either.
Definition 5. Preferences form an odd GGSC cycle on $\mathcal{S}$ if preferences form a GGSC cycle on $\mathcal{S}$ and there is an odd $K > 2$.

Hence, preferences of the professor, the student, and the post-doc over the clothes which we described in the Introduction and illustrated formally in Example 2 form an odd GGSC cycle with $M = K = 3$.

3 Main result

We now present the main result of this paper.

Theorem 1. Suppose that in the trading economy $\mathcal{E} = \{\Omega, (u_i, i \in I)\}$ agents’ preferences satisfy GGSC on $\mathcal{S}$ and do not form any odd GGSC cycles. Then this trading economy has a competitive equilibrium.

Proof. Take any element in $\mathcal{S} = (S_1, ..., S_M)$. Consider a graph $G$, whose nodes are the elements of $\mathcal{S}$ and whose edges represent the fact that some agents have GSC preferences between these two nodes. Since $\mathcal{S}$ does not form an odd cycle, it is a bipartite graph whose nodes can be divided into two disjoint sets $S^A$ and $S^B$ such that every edge connects a node in $S^A$ to the node(s) in $S^B$ only. Consider any agent $i \in I$. Without loss of generality, assume agent $i$ has GSC preferences over $S_m$ and $S_{m+1}$ and that $S_m$ is a node in $S^A$ (call it $S_m^A$) and $S_{m+1}$ is a node in $S^B$ (call it $S_{m+1}^B$ for symmetry). Consider the valuation function $v_i(T) = u_i(T \cap (S_m^A \cup S_{m+1}^B))$ for any goods in $T \subseteq S^A \cup S^B$. This induces an economy $\mathcal{E}^* = \{\Omega, (v_i, i \in I)\}$. Agent $i$’s preferences described by utility function $V_i(T)$ satisfy GSC over $S^A$ and $S^B$ in $\mathcal{E}^*$. Hence, the utility function of every $i \in I$ in $\mathcal{E}^*$ satisfies GSC over $S^A$ and $S^B$. Therefore, following Sun and Yang (2006) the general trading economy on graph $G$ must have at least one competitive equilibrium.

4 Auction

Sun and Yang (2008, 2009) introduced a tâtonnement process – called a dynamic double-track (DDT) procedure – that finds the competitive equilibrium allocation and prices when agents’ preferences satisfy GSC. We can immediately extend this to GGSC. First, we need to define a new partial order on the lattice of prices $p \leq^* q$ if and only if $p_\omega \leq q_\omega$ for $\omega \in S_i \in S^A$ and $p_\omega \geq q_\omega$ for $\omega \in S_j \in S^B$. Hence, we say that price vector $q$ is greater than price vector $p$ with respect to $\leq^*$ if and only if $q$ is greater for goods in $S^A$ and smaller for goods in $S^B$ in the induced economy $\mathcal{E}^*$. We label the DDT procedure and the global DDT (GDDT) procedure defined in Sun and Yang (2009) as DDT$^*$ and GDDT$^*$ when applied to $\mathcal{E}^*$ respectively.
Proposition 1. Suppose that $u_{i} : 2^{\Omega} \rightarrow Z_{+}$ (bounded from above) and preferences of all agents satisfy GGSC without odd cycles. Then

- starting at the lowest price vector (w.r.t. $\leq^{*}$) DDT$^{*}$ procedure converges to the lowest equilibrium price vector (w.r.t. $\leq^{*}$) in a finite number of rounds.

- starting at any price vector, the GDDT$^{*}$ procedure converges to an equilibrium price vector in a finite number of rounds.

Proof. We reduce $\mathcal{E}$ to $\mathcal{E}^{*}$ and apply the DDT or the GDDT procedure over $S^{A}$ and $S^{B}$.

The processes are simple and intuitive and we have already illustrated it in Example 2. For example, in the DDT$^{*}$ procedure, the seller starts off with low prices for goods in $S^{A}$ and with high prices in $S^{B}$. This is only possible whenever there are no odd GGSC cycles. At every stage of the process, the seller asks agents to report their demands. The prices of over-demanded goods are increased and prices of under-demanded goods are decreased. Whenever there are odd GGSC cycles, the seller would have to start off the auction with high (low) prices for goods regarded as complements by some agents and the descending (ascending) auction may not find the market clearing prices (Sun and Yang, 2008, Section 3.2). Similarly, we can immediately adapt the strategy-proof and efficient dynamic mechanism based on the DDT procedure for preferences that satisfy GGSC without odd cycles (Sun and Yang, 2008, Section 4).

One application of this auction is the allocation of airport landing and take-off slots (Rassenti et al., 1982). Suppose an airport has (at least) two runways so that one plane can land and one plane can take off within a given five-minute slot. Airlines regard different pairs of landing and take-off slots as complements depending on how long they need to clean, refuel and refill the aircraft with passengers. However, any nearby landing or takeoff slots would be substitutes for all airlines. Therefore, the airline preferences will satisfy GGSC preferences without odd cycles and an auction based on the DDT$^{*}$ procedure can be run. Note that we do not even require every airline to bid for both landing and take-off slots since the GGSC preferences without odd cycles are not violated by airlines that only require either a landing or a take-off slot.

5 Extensions

5.1 Trading networks

Hatfield et al. (2013) consider a model of trading networks in which agents’ preferences
are **fully substitutable** over the contracts (which specify trades and their prices) that they are involved in. Full substitutability means that any two upstream contracts (i.e. those in which the agent is a buyer) or any two downstream contracts (i.e. those in which the agent is a seller) are substitutes, but any upstream and any downstream contract are complements for any agent. They show that competitive equilibrium (with and without personalized prices) exists in their model in arbitrary trading networks.\(^7\) Drexl (2013) generalized this model to the case of *full substitutes and complements* by allowing agents to view both their upstream and downstream contracts as gross substitutes and complements. Our results for the trading economy show that this can be made even more general – we can allow a GGSC structure with no odd cycles for both upstream and downstream contracts. Moreover, the properties of competitive equilibria in trading networks described Hatfield et al. (2013) continue to hold whenever preferences have a GGSC structure with no odd cycles. In particular, the allocation supported by a competitive equilibrium is *stable* (Theorem 5 in Hatfield et al. (2013)).\(^8\)

5.2 Multi-unit environments

We can extend the GGSC model to an environment where there are multiple units of each good (Shioura and Yang, 2013). Suppose there are \(z_\omega\) units of each good \(\omega \in \Omega\) available in the economy, so \(z\) is a vector of overall endowment of goods, and every unit of any given good is perfectly substitutable. Goods are still partitioned into \(M\) sets of imperfectly substitutable goods. Then the set of all bundles of goods is \(Z = \{x \in \mathbb{Z}_{+}^{[\Omega]} | x \leq z\}\). Each agent has a valuation function \(u_i : Z \rightarrow \mathbb{R}\) and a net utility function \(U_i([\Pi; p]) \equiv u_i(z_\omega) - p_\omega \cdot z_\omega\) for any \(z_\omega \in Z\) and any partition \(\Pi\) on \(Z\). The rest of the definitions can be extended accordingly and the proof of Theorem 1 shows that the multi-unit version of GGSC can be reduced to the setting of Shioura and Yang (2013) whenever there are no odd GGSC cycles in agents’ preferences.

\(^7\)In a footnote on p. 18, Hatfield et al. (2013) say

Note that our network setting is clearly more general, as it cannot be embedded in the setting of Sun and Yang (2006). E.g., consider a simple market with three agents \(i, j, \) and \(k\), where \(i\) can sell trades to both \(j\) and \(k\), \(j\) can sell trades only to \(k\), \(k\) cannot sell trades to anyone, and all agents’ preferences are fully substitutable. In this market, it is impossible to separate trades into two groups \(S_1\) and \(S_2\) in such a way that every agent views trades in one group as substitutes and views trades in different groups as complements.

This reasoning is somewhat misleading. Rather than having preferences over contracts, agents have preferences over roles (buyer or seller) that they play in these contracts. In fact, since preferences are quasilinear, we can separate the possible downstream trades for each agent into \(S_1\) and the possible upstream trades for each agent into \(S_2\) and recover the Sun and Yang’s framework.

\(^8\)Teytelboym (2013) shows that the assumption of quasilinearity is crucial to these results.
6 Conclusion

We showed that competitive equilibrium exists in an exchange economy when agents’ preferences have a particular structure: generalized gross substitutes and complements with no odd cycles. We showed that we can find these competitive equilibria using a simple tâtonnement process. The results of this paper can be used in many market design applications, such as auctions and matching markets. A simple procedure that finds competitive equilibria in exchange economies with indivisible goods whenever they exist remains an open problem.
Absence of odd GGSC cycles is not necessary for the existence of a competitive equilibrium  We claimed that absence of odd GGSC cycles is only a sufficient condition for stability. As our example above showed, there are valuations for agents with the same preferences such that no competitive equilibrium in the economy exists. The example illustrated in Table 3 below shows that lack of odd cycles GGSC is not necessary for equilibrium existence.

<table>
<thead>
<tr>
<th>Ψ</th>
<th>$u_i(Ψ_i)$</th>
<th>$u_j(Ψ_j)$</th>
<th>$u_k(Ψ_k)$</th>
</tr>
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<tbody>
<tr>
<td>∅</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>{ω_1}</td>
<td>2</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>{ω_2}</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>{ω_3}</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>{ω_1,ω_2}</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>{ω_2,ω_3}</td>
<td>0</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>{ω_3,ω_1}</td>
<td>0</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>{ω_1,ω_2,ω_3}</td>
<td>5</td>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

Price vector $p^* = (4.5, 2, 4.5)$ supports a competitive equilibrium allocation $Π^* = \{\{Ψ_i, Ψ_j, Ψ_k\} = \{\{ω_2\}, ∅, \{ω_1,ω_3\}\}$. We find this by inspection since we cannot apply the modified double-track process. The following proposition is trivial to prove and follows from our examples in Section 2.3.

**Proposition 2.** For any trading economy $E = \{Ω, (u_i, i ∈ I)\}$, if agents’ preferences satisfy GGSC on $S$ and form odd GGSC cycles, there exist agents’ valuations over bundles with the same GGSC structure on $S$ such that no competitive equilibrium exists.
References


